A Guide to the Proof of Fujita's Freeness Conjecture

How can sections of mL help us find a section of $K_X + 3L$ nonzero at a given point?

Personal reading notes · Edited by Minzhe Zhu · November 2026

We begin with a smooth projective surface. The lattice count then involves only two variables, and a curve is the only positive-dimensional center we have to rule out. These simpler calculations bring out the main ideas. The final part explains exactly which steps depend on the dimension and how they work in general.

Main theorem (Theorem 1.1 of the source paper)

Let X be a smooth connected complex projective variety of dimension $n \geq 1$, and let L be an ample line bundle on X. Then the line bundle

$$K_X + (n+1)L$$

is globally generated.

Corollary 6.3 For every integer $k \geq n+1$, the line bundle $K_X + kL$ is globally generated.

These notes explain the proof in OpenAI's *Fujita's freeness conjecture* (September 23, 2026) and use its theorem numbers. We write line bundles additively. Put $P = K_X + (n+1)L$, and write $P_x = P \otimes \mathbb{C}(x)$ for its one-dimensional fiber at x. Global generation means that for each closed point x there is a section $s \in H^0(X, P)$ with $s(x) \neq 0$. The constant $n+1$ is sharp over all pairs (X, L). For $(\mathbb{P}^n, \mathcal{O}(1))$ with hyperplane class H, we have $K_{\mathbb{P}^n} + (n+1)H = 0$, whereas $K_{\mathbb{P}^n} + nH = -H$ has no nonzero sections.

Suppose x is a base point of $|K_X + 3L|$

Then every global section of $P = K_X + 3L$ vanishes at x. We start by looking at a different set of sections. For a nonzero $s \in H^0(X, mL)$, its order of vanishing at x is $\mathrm{ord}_x(s)$. After we choose a local trivialization near x for mL, this is the lowest total degree of a nonzero term in the Taylor expansion of s. Section 3 studies the average of these orders in an adapted basis.

Section 6 constructs an effective rational divisor B on a smooth birational model $\pi : Y \to X$ and a nonempty reduced divisor S mapping to x. The coefficient of B along each component $F \subset S$ equals its log discrepancy $A_X(F)$. Every other component meeting S satisfies $\mathrm{coeff}_F(B) < A_X(F)$. These coefficients let us construct a section on S, lift it to Y by vanishing, and descend it to X. The resulting section of P is nonzero at x.

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Section 2

Toolbox

Throughout, write $R_j = H^0(X, jL)$ and $N_j = \dim R_j$. Since L is ample, its Hilbert polynomial gives

$$N_j = \frac{L^n}{n!} j^n + O(j^{n-1}), \qquad L^n \in \mathbb{Z}_{\geq 1}.$$

For a prime divisor F on a smooth birational model $\pi : Y \to X$, write $A(F) = 1 + \operatorname{ord}_F(K_{Y/X})$, where $K_{Y/X} = K_Y - \pi^* K_X$. We use $c_X(v)$ for the closure of the center of a valuation. The condition $x \in c_X(v)$ thus allows the center to be a curve passing through x.

2.2 SNC retraction

On a common log resolution of a collection of effective divisors D_i, the exceptional divisors and the strict transforms of their components form a simple normal crossings (SNC) boundary $\sum_j E_j$. Let w be a positive multiple of a divisorial valuation and set $u_j = w(E_j)$. If some $u_j > 0$, these weights define a monomial valuation r with log discrepancy $A(r) = \sum_j u_j A(E_j)$. Lemma 2.2 gives

$$r(D_i) = w(D_i), \qquad A(r) \leq A(w), \qquad c_X(r) \supseteq c_X(w).$$

The equality follows because each divisor has a monomial local equation up to a unit. The discrepancy inequality follows because the reduced SNC pair $(Y, \sum E_j)$ is log canonical (lc). For a boundary B on a smooth variety, lc means that $\operatorname{ord}_F(B) \leq A(F)$ for every prime divisor over the variety. Strict inequality for every such divisor means that the pair is Kawamata log terminal (klt).

2.3 Lower semicontinuity of the log canonical threshold

Write $\operatorname{lct}_x(\mathfrak{a})$ for the log canonical threshold of an ideal $\mathfrak{a}$ at x. It equals $\min_{x \in c_X(F)} A(F)/\operatorname{ord}_F(\mathfrak{a})$, where only positive denominators are considered. When finite, this minimum is attained on a log resolution. For the unit ideal $\mathfrak{a} = \mathcal{O}_X$, we have $\operatorname{lct}_x(\mathfrak{a}) = +\infty$. In an algebraic family of ideals $\mathfrak{a}_s$, the set $\{(x, s) : \operatorname{lct}_x(\mathfrak{a}_s) \leq c\}$ is Zariski closed. Section 4 uses this to turn the existence of a valuation into a closed condition on a flag variety.

2.4 Kawamata–Viehweg vanishing

Let Y be smooth and projective, D an integral divisor, and Θ an effective rational SNC boundary with coefficients in $[0, 1)$. Suppose $D - (K_Y + \Theta)$ is nef and big, meaning it has nonnegative intersection with every curve and positive volume. Then $H^i(Y, \mathcal{O}_Y(D)) = 0$ for every $i > 0$. Sections 3 and 6 take Θ to be the fractional part of a rational divisor and use $H^1 = 0$ to lift sections.

2.5 Descending a section from Y to X

Let C be an integral divisor on Y whose positive part is supported on π-exceptional divisors. Every section of $\pi^* P + C$ descends to X as a regular section s of P. Suppose F maps to x and C has coefficient zero along F. If the section upstairs is nonzero at the generic point of F, then $s(x) \neq 0$. This uses normality of X and the fact that a local function vanishing at x has positive order along F.

Section 3 Suppose x is a base point of $|K_X + 3L|$

How much do sections of jL vanish at x on average?

Unless stated otherwise, X is now a smooth complex projective surface, L is an ample line bundle on X, and $x \in X$ is a fixed closed point. Set $P = K_X + 3L$. Choose a basis of $H^0(X, jL)$ adapted to ord_x. Average the orders of its sections at x and divide by j. Proposition 3.2 says that if x is a base point of $|P|$, the lower limit of these numbers as $j \to \infty$ is strictly greater than $2/3$.

Step 1. Choose regular parameters u, w at x and a local frame for L. Its powers trivialize jL. In the Taylor expansion of a section, take a term of smallest total degree. Break ties by taking the smallest exponent of u. If the resulting term is $cu^\alpha w^\beta$, write $\mathrm{in}(s) = (\alpha, \beta)$. Then $\mathrm{ord}_x(s) = \mathrm{mult}_x(s) = \alpha + \beta$. Put

$$H_j = \{\mathrm{in}(s) : 0 \neq s \in R_j\} \subset \mathbb{Z}^2_{\geq 0}.$$

By eliminating terms with the same initial exponent, we obtain a basis with distinct initial exponents. In any nonzero linear combination, the smallest initial term with a nonzero coefficient survives. Hence $\#H_j = N_j$. Initial exponents add under multiplication, so $H_j + H_h \subseteq H_{j+h}$. This basis is adapted to the filtration by ord_x. The dimensions of the filtration spaces determine how often each order occurs. The normalized mean is

$$\mu_j = \frac{1}{jN_j} \sum_{(\alpha,\beta)\in H_j} (\alpha + \beta).$$

Choose $k > 0$ so that kL is very ample. A general member C of $|kL|$ through x is smooth at x and shares no component with $\mathrm{div}(s)$. It follows that

$$\mathrm{mult}_x(s) \leq C \cdot \mathrm{div}(s) = (kL) \cdot (jL) = kjL^2.$$

This linear bound holds for every nonzero section. We will use it again in Section 4.

To bound μ_j from below, place N_j distinct lattice points in the closed first quadrant. Their total degrees have the smallest possible sum if we fill the layers in order of increasing degree. The layer of degree d contains $d + 1$ points. Filling every layer through degree r gives mean degree

$$\frac{\sum_{d=0}^{r} d(d+1)}{\sum_{d=0}^{r}(d+1)} = \frac{2r}{3}.$$

Since $N_j = L^2j^2/2 + O(j)$, the last layer needed has degree $r = \sqrt{L^2}j + O(1)$. Filling only part of that layer changes the average by $O(1)$. Thus $\liminf_j \mu_j \geq 2\sqrt{L^2}/3$.

Proposition 3.2 for surfaces If $x \in \mathrm{Bs}\,|K_X + 3L|$, then $\liminf_{j\to\infty} \mu_j > 2/3$.

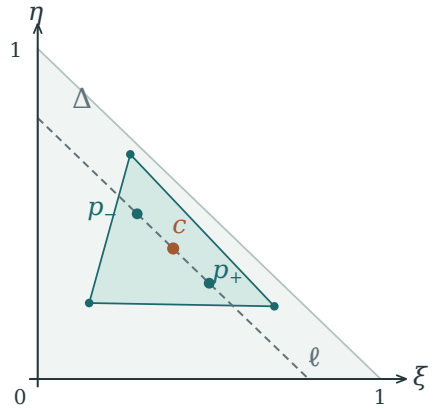
If $L^2 > 1$, the count already gives the strict inequality. The only case left is $L^2 = 1$ with $\mu_j \to 2/3$ along a subsequence. From this equality we will construct a section of P that is nonzero at x.

Step 2. Suppose equality holds. We will use the rescaled lattice points $j^{-1}H_j$ to construct two sections. Consider the unit triangle

$$\Delta = \{(\xi, \eta) \in \mathbb{R}^2_{\geq 0} : \xi + \eta \leq 1\}$$

and fix a small disk whose closure lies in its interior. Suppose the rescaled points miss the disk. Compared with the filling of least total degree, we must then replace at least $c_0 j^2$ points of degree $\leq (1-\delta)j$. Each new point has degree at least $j - O(1)$, and the other replacements cannot lower the sum. The increase is at least a positive constant times j^3. After division by jN_j there is still a fixed positive gap, contradicting $\mu_j \to 2/3$. Thus every such disk contains a rescaled point for all sufficiently large j in the subsequence.

Choose a rational number $2 < t' < 3$. For some positive integer h, we want sections of $ht'L$ with multiplicity $2h$ at x. Their normalized initial exponents must therefore lie on the line $\ell = \{\xi+\eta = 2/t'\}$. It passes through the interior point $c = (1/t', 1/t')$ of Δ. The inequality $t' < 3$ leaves the positivity we will need for Kawamata–Viehweg vanishing.



Choose three small disks around c so that a triangle formed by picking one point from each always contains c in its interior. In **one and the same degree** j, choose $q_i = \gamma_i/j$ in each disk, with $\gamma_i \in H_j$. Intersect their convex hull with ℓ. On this segment choose rational points p_-, p_+ on opposite sides of c, ordered by their first coordinate.

To realize these two rational points by sections, write $p_\pm = \sum_i b_{\pm i} q_i$ with $b_{\pm i} \in \mathbb{Q}_{\geq 0}$ and $\sum_i b_{\pm i} = 1$. Take a common multiple M so that all $Mb_{\pm i}$ and $h = jM/t'$ are integers. Choose sections with $\mathrm{in}(s_i) = \gamma_i$. The products $r_\pm = \prod_i s_i^{Mb_{\pm i}}$ lie in $R_{ht'}$ and have initial exponents $jMp_\pm$. Thus

$$\mathrm{in}(r_\pm) = (a_\pm, 2h - a_\pm), \qquad a_- < h < a_+.$$

Both sections have multiplicity exactly $2h$ at x. After we blow up x, the inequalities $a_- < h < a_+$ will bound their common order of vanishing at every point of the exceptional curve.

Step 3. Blow up x. Write $\pi : Y = \mathrm{Bl}_x X \to X$ for the blowup and $E \simeq \mathbb{P}^1$ for its exceptional curve. Pull back $r_\pm$, remove the common divisor $2hE$, and restrict to E. We get their lowest-degree homogeneous forms $g_\pm$, both of degree $2h$. For instance, on the chart $u = zw$, divide by w^{2h} and then set $w = 0$. What remains is $g_\pm(z, 1)$.

Each point of E represents a tangent direction at x. At $z_0 = [0 : 1]$ we have $\mathrm{ord}_{z_0}(g_\pm) = a_\pm$, so g_- has order less than h. On the rest of E, the form g_+ has only $2h - a_+ < h$ zeros left, counted with multiplicity. Thus the restricted ideal $\mathfrak{b}_E = (g_-, g_+)$ satisfies

$$\mathrm{ord}_z(\mathfrak{b}_E) < h \quad (z \in E), \qquad \mathrm{lct}_z(E, \mathfrak{b}_E) > 1/h.$$

The two forms can still have common zeros. At each of them, however, the common multiplicity multiplied by $1/h$ rounds down to zero.

To lift a section from the exceptional curve, first check the positivity needed for vanishing. Put $H = \pi^*(ht'L) - 2hE$ and let $\mathfrak{b}$ be the base ideal of the two sections after removing $2hE$. Since $K_Y = \pi^* K_X + E$, we have

$$\pi^* P - (K_Y + E) = H/h + (3 - t')\pi^* L.$$

Take a projective log resolution $\sigma : T \to Y$ of $(Y, E, \mathfrak{b})$. Let E' be the strict transform of E and write $\mathfrak{b}\mathcal{O}_T = \mathcal{O}_T(-G)$. Set $D_T = K_T + \sigma^*(\pi^* P - K_Y - E) - \lfloor G/h \rfloor$. Then

$$D_T - K_T - \{G/h\} = (3 - t')(\pi\sigma)^* L + (\sigma^* H - G)/h.$$

The first term is nef and big. The second is nef because $\sigma^* H - G$ is globally generated. Thus $H^1(T, \mathcal{O}_T(D_T)) = 0$. The exact sequence for E' now gives a surjection

$$H^0(T, D_T + E') \longrightarrow H^0(E', (D_T + E')|_{E'}).$$

What section can we take on the right? The map $E' \to E$ is a birational morphism of smooth curves, hence an isomorphism. The divisor G does not contain E', and its restriction gives the divisor of zeros of $\mathfrak{b}_E$. The SNC condition lets us round down before or after restricting. By adjunction, the restricted line bundle is isomorphic to $P_x \otimes \mathcal{O}_{E'}(-\lfloor G|_{E'}/h \rfloor) = P_x \otimes \mathcal{O}_{E'}$. Take any $0 \neq e \in P_x$ and lift the resulting constant section on E' to T. The correction divisor relative to $(\pi\sigma)^* P$ is

$$K_{T/Y} - \sigma^* E + E' - \lfloor G/h \rfloor.$$

Its coefficient along E' is zero and its positive part is exceptional. Lemma 2.5 now gives $s(x) \neq 0$. This rules out equality when x is a base point and proves Proposition 3.2.

Section 4 Minimization in a fixed degree

Why does the exponential average attain its minimum?

Continue to assume that x is a base point of $|P|$. Fix a degree m and a real number $t > 0$, and put $N = N_m > 0$. Vary both an ordered basis $\mathbf{s} = (s_1, \dots, s_N)$ of R_m and a nonzero monomial valuation v. Consider the exponential average and its infimum

$$F_{m,t}(v, \mathbf{s}) = \frac{1}{N} \sum_{i=1}^{N} \exp\Big(\frac{A(v)}{t} - \frac{v(s_i)}{m}\Big), \qquad f_{m,t}(x) = \inf_{\substack{\mathbf{s}, v \neq 0 \\ x \in c_X(v)}} F_{m,t}(v, \mathbf{s}).$$

Our goal is to show that the infimum is attained. There are m_0 and $t_0 \in (0, 3)$ such that for every $m \geq m_0$ and $t \in [t_0, 3]$ some pair $(v, \mathbf{s})$ satisfies $F_{m,t}(v, \mathbf{s}) = f_{m,t}(x)$. Proposition 4.4 also gives bounds $0 < a_* \leq A(v) \leq a^* < \infty$ for every minimizing pair, independent of m, t. At this stage we only require $x \in c_X(v)$. The center could still be a curve through x.

Why minimize this function? Its derivative supplies coefficients for an effective divisor. Write $D_i = \operatorname{div}(s_i)$, so $v(s_i) = v(D_i)$. Hold the basis $\mathbf{s}$ fixed and differentiate $\log F_{m,t}$ along the ray λv. Dividing each exponential factor by the sum of all the factors gives the weights $p_i(\lambda)$ in the derivative.

$$\frac{d}{d\lambda} \log F_{m,t}(\lambda v, \mathbf{s}) = \frac{A(v)}{t} - \sum_i p_i(\lambda) \frac{v(D_i)}{m}, \qquad p_i(\lambda) = \frac{e^{-\lambda v(D_i)/m}}{\sum_h e^{-\lambda v(D_h)/m}}.$$

These weights are positive and sum to 1. If $(v, \mathbf{s})$ is a minimizing pair, the derivative at $\lambda = 1$ vanishes. The weighted mean of the normalized orders $v(D_i)/m$ is then exactly $A(v)/t$. Use these weights $p_i = p_i(1)$ as coefficients for the zero divisors $D_i \sim mL$, with the indicated scale factor. This gives the effective $\mathbb{R}$-divisor

$$\Delta = \frac{t}{m} \sum_i p_i D_i \sim_{\mathbb{R}} tL, \qquad v(\Delta) = \frac{t}{m} \sum_i p_i v(D_i) = A(v).$$

Section 5 will choose $t \in (0, 3)$ and a sequence of degrees $m_\nu \to \infty$ such that every pair $(v, \mathbf{s})$ attaining $f_{m_\nu,t}(x)$ has $c_X(v) = \{x\}$. Section 6 will replace the corresponding Δ by a rational approximation and perturb it. This produces a boundary on a model $Y \to X$ and a reduced divisor S mapping to x. The boundary has coefficient 1 on each component of S and coefficient strictly less than 1 on every other component meeting S. We will lift a section from S.

For the existence proof, start with a sequence $(v_j, \mathbf{s}_j)$ along which $F_{m,t}$ approaches its infimum. First bound the scales $A(v_j)$. Then use a full flag to fix a single basis. Once the basis is fixed, resolve the zero divisors of its sections together and take a limit of valuation weights on that one model.

Step 1. First find a test pair with $F_{m,t} < 1$. Choose a basis $\mathbf{s}$ adapted to ord_x. The exceptional curve of the blowup of x gives ord_x, so $A(\operatorname{ord}_x) = 2$. The function $F_{m,t}(\lambda \operatorname{ord}_x, \mathbf{s})$ extends continuously to $\lambda = 0$ with value 1. All its weights there are $1/N$, so its right derivative is

$$\frac{d}{d\lambda} F_{m,t}(\lambda \operatorname{ord}_x, \mathbf{s})\big|_{0+} = \frac{2}{t} - \mu_m.$$

By Proposition 3.2 we can choose $t_0 < 3$ close enough to 3 that the derivative has a uniform negative upper bound for $m \geq m_0$ and $t \in [t_0, 3]$. Since $\operatorname{mult}_x(s) \leq Cm$, the second-order remainder is uniformly $O(\lambda^2)$. One small $\lambda > 0$ therefore gives $f_{m,t}(x) \leq 1 - \delta$ for a fixed $\delta > 0$.

Now consider a test pair satisfying $F_{m,t} \leq 1 - \delta/2$ and write $a = A(v)$. Lemma 4.1 uses $\operatorname{lct}_x(\operatorname{div}(s)) \geq 1/\operatorname{mult}_x(s)$ to obtain $v(s) \leq a \operatorname{mult}_x(s)$. Consequently $F_{m,t} \geq e^{-Ca}$. On the sublevel set $F_{m,t} \leq 1 - \delta/2$, this bounds a away from zero.

To bound $a = A(v)$ from above, choose an integer $k \geq 6$ such that kL is very ample. Take $\sigma_0, \sigma_1, \sigma_2 \in R_k$ with $\sigma_0(x) \neq 0$ such that $z_1 = \sigma_1/\sigma_0$ and $z_2 = \sigma_2/\sigma_0$ are local parameters at x. For each residue class r modulo k, fix a positive integer $b_r \equiv r \pmod{k}$ and a section $\tau_r \in R_{b_r}$ nonvanishing at x. Write $m = qk + b_r$. For $i + j \leq q$, multiplying the local monomial $z_1^i z_2^j$ by $\tau_r \sigma_0^q$ gives a global section. Thus R_m maps onto the space of jets of order q at x. Since $q \sim m/k$, the ratio of its dimension $\binom{q+2}{2}$ to N_m has a positive lower bound κ.

In any basis, at least κN_m sections have a nonzero jet of order q. Each satisfies $\mathrm{mult}_x(s_i) \leq q$. We chose $k \geq 6$ so that, for every $t \in [t_0, 3]$,

$$q = \frac{m - b_r}{k} \leq \frac{m}{6} \leq \frac{m}{2t}.$$

By Lemma 4.1, these sections satisfy $v(s_i) \leq aq \leq ma/(2t)$. Their contribution alone gives $F_{m,t} \geq \kappa e^{a/(2t)}$. Combining the bounds, we obtain

$$\kappa \leq f_{m,t}(x) \leq 1 - \delta, \qquad 0 < a_* \leq A(v) \leq a^* < \infty \quad (F_{m,t} \leq 1 - \delta/2).$$

The constants work for every sufficiently large m and throughout the indicated interval of t. In Section 5 we will need a_* before we choose t.

Step 2. Fix m, t in the range of Step 1. We want a single basis for the minimizing sequence. Rational weights approximate real weights, so we can take test pairs whose valuations are positive multiples of divisorial valuations and whose values approach $f_{m,t}(x)$. Order each basis by decreasing normalized value. The scale bounds and $v(s_i)/A(v) \leq Cm$ let us pass to a subsequence on which

$$A(v_\nu) \to a > 0, \qquad \frac{v_\nu(s_{\nu,i})}{A(v_\nu)} \to q_i, \qquad q_1 \geq \cdots \geq q_N \geq 0.$$

It follows that $f_{m,t}(x) = N^{-1} \sum_i \exp(a/t - aq_i/m)$. At least one $q_i > 0$, since otherwise the right side would exceed 1. It is enough to realize all the lower bounds q_i at once on one fixed basis. Increasing a section's value decreases its contribution to the exponential average.

Fix rational numbers $\alpha_1 \geq \cdots \geq \alpha_N \geq 0$. A basis determines the full flag $U_i = \langle s_1, \ldots, s_i \rangle$, with $\dim U_i = i$. The flag variety $\mathrm{Flag}(R_m)$ parametrizes these flags. Write $\mathfrak{b}_i$ for the base ideal of U_i, locally generated by $s_1, \ldots, s_i$. For any valuation w we have $w(\mathfrak{b}_i) = \min_{h \leq i} w(s_h)$. Since the α_i are in decreasing order,

$$w(s_i) \geq \alpha_i A(w) \text{ for every } i \quad \Longleftrightarrow \quad w(\mathfrak{b}_i) \geq \alpha_i A(w) \text{ for every } i.$$

The condition on the right depends only on the flag. If every α_i is zero, then at any closed point z we can take $w = \mathrm{ord}_z$. If some $\alpha_i > 0$, choose a positive integer M so that M/α_i is an integer for each positive entry, and put

$$\mathfrak{a}_{\alpha, U_\bullet} = \sum_{\alpha_i > 0} \mathfrak{b}_i^{M/\alpha_i}.$$

A valuation takes the minimum on a sum of ideals. The bounds above are therefore equivalent to $w(\mathfrak{a}_{\alpha, U_\bullet}) \geq MA(w)$. There is a divisorial valuation w whose center contains z and which meets all the bounds if and only if

$$\mathrm{lct}_z(\mathfrak{a}_{\alpha, U_\bullet}) \leq 1/M.$$

By Lemma 2.3 these pairs $(z, U_\bullet)$ form a closed subset of $X \times \mathrm{Flag}(R_m)$. Fixing $z = x$ gives a closed subset of the flag variety. Write Z_α for the projection of the locus of pairs to X. The flag variety is projective, so Z_α is closed as well.

Choose ordered rational vectors $\alpha^{(k)}$ increasing coordinatewise to (q_i), with $\alpha_i^{(k)} < q_i$ whenever $q_i > 0$. Fix k. All sufficiently late terms of the minimizing sequence satisfy the corresponding bounds, so the associated closed set of flags is nonempty. These closed sets are nested. Projectivity gives a flag common to all of them. Fix a basis $\mathbf{s}$ adapted to it. For each k we can now choose a divisorial valuation w_k whose center contains x and such that $w_k(s_i)/A(w_k) \geq \alpha_i^{(k)}$ for every i. The valuation w_k may vary with k. The **sections are the same** throughout.

Step 3. Put $D_i = \mathrm{div}(s_i)$ and take a common projective log resolution $\pi : Y \to X$, including the exceptional curves in the SNC divisor. Write E_j for its components and set $a_j = A(E_j) > 0$ and $d_{ji} = \mathrm{ord}_{E_j}(D_i)$. Let $r_k = r_Y(w_k)$ be the SNC retraction of w_k on Y. Lemma 2.2 gives $r_k(s_i) = w_k(s_i)$ and $A(r_k) \le A(w_k)$, and the center still contains x. For large k some $\alpha_i^{(k)} > 0$, so $r_k(s_i) > 0$ and hence $r_k \neq 0$. With the basis $\mathbf{s}$ fixed, we get

$$F_{m,t}(r_k, \mathbf{s}) = \exp\left(\frac{A(r_k) - A(w_k)}{t}\right) F_{m,t}(w_k, \mathbf{s}) \le F_{m,t}(w_k, \mathbf{s}).$$

We may therefore replace w_k by r_k. Normalize r_k to $r_k/A(r_k)$. The section values still satisfy $r_k(s_i)/A(r_k) \ge \alpha_i^{(k)}$.

The SNC divisor has finitely many strata. After passing to a subsequence, all the r_k belong to the same irreducible component of one stratum. If just one divisor occurs, there is a unique weight with log discrepancy 1. At a crossing of E_1, E_2, the normalized weights satisfy $u_1, u_2 \ge 0$ and $a_1u_1 + a_2u_2 = 1$. Write $\theta = a_1u_1$. The value of the i-th section is

$$\theta\frac{d_{1i}}{a_1} + (1-\theta)\frac{d_{2i}}{a_2}, \qquad 0 \le \theta \le 1.$$

If θ reaches an endpoint, the center on Y expands from the crossing point to a curve whose image still contains x. A convergent subsequence gives a valuation with $A(\hat{v}) = 1$ and $\hat{v}(s_i) \ge q_i$. Restore the scale by setting $v = a\hat{v}$. Then

$$f_{m,t}(x) \le F_{m,t}(v, \mathbf{s}) \le \frac{1}{N}\sum_i \exp\left(a/t - aq_i/m\right) = f_{m,t}(x).$$

This proves Proposition 4.4. The resulting real-weight monomial valuation and our fixed basis attain the infimum.

Lemma 4.5 compares x with a very general point y of an irreducible curve W through x. Fix m. Recall that $z \in Z_\alpha$ means there are a basis and a divisorial valuation whose center contains z and whose normalized values are at least α_i in each coordinate. If $W \not\subseteq Z_\alpha$, then $W \cap Z_\alpha$ is finite. There are only countably many rational vectors α. Over the complex numbers we can choose a smooth closed point $y \in W$ outside all these finite sets. Now $y \in Z_\alpha$ implies $W \subseteq Z_\alpha$ and, in particular, $x \in Z_\alpha$.

Take any divisorial test pair $(v, \mathbf{s})$ at y. Put $a = A(v)$ and order the basis by decreasing $\beta_i = v(s_i)/a$. Choose rational numbers $\alpha_i^{(\ell)} \uparrow \beta_i$ so that each list satisfies $\alpha_1^{(\ell)} \ge \cdots \ge \alpha_N^{(\ell)} \ge 0$. The pair $(v, \mathbf{s})$ itself meets the ℓ-th set of bounds. Thus $y \in Z_{\alpha^{(\ell)}}$, and hence $x \in Z_{\alpha^{(\ell)}}$.

For each ℓ we can therefore choose a basis $\mathbf{s}^{(\ell)}$ and a divisorial valuation w_ℓ at x with $w_\ell(s_i^{(\ell)})/A(w_\ell) \ge \alpha_i^{(\ell)}$. To compare the two values of $F_{m,t}$, rescale by setting $w'_\ell = aw_\ell/A(w_\ell)$. The two tests now have the same log discrepancy a, hence the same first term a/t in the exponent. Since $w'_\ell(s_i^{(\ell)}) \ge a\alpha_i^{(\ell)}$ and the section value enters with a minus sign, we obtain

$$f_{m,t}(x) \le F_{m,t}(w'_\ell, \mathbf{s}^{(\ell)}) \le \frac{1}{N}\sum_i \exp\left(\frac{a}{t} - \frac{a\alpha_i^{(\ell)}}{m}\right).$$

As $\ell \to \infty$, the finite sum on the right tends to $F_{m,t}(v, \mathbf{s})$. This holds for every divisorial test pair at y. By the rational approximation in Step 2, the infimum over these tests is $f_{m,t}(y)$. Hence $f_{m,t}(x) \le f_{m,t}(y)$. The closed sets used to choose y are independent of t, so the same y works for every $t > 0$.

Finally, suppose $(v_0, \mathbf{s}_0)$ attains the minimum at x and $c_X(v_0) = W$. Since $y \in W$, the same pair is admissible at y, and $F_{m,t}(v_0, \mathbf{s}_0)$ depends only on the valuation and the basis. Thus

$$f_{m,t}(y) \le F_{m,t}(v_0, \mathbf{s}_0) = f_{m,t}(x) \le f_{m,t}(y).$$

The same pair still minimizes at y. Section 5 can now perturb the valuation in the tangent direction of W at the smooth point y and compare the perturbed value with this minimum.

Section 5 Ruling out a curve center

Why must a minimizing valuation be centered at a point?

Proposition 5.1 Suppose x is a base point of $|P|$. There exist $0 < t < 3$ and a sequence of integers m_ν tending to infinity such that every $f_{m_\nu,t}(x)$ is attained. In every minimizing pair for any of these degrees, the valuation has center $\{x\}$.

Step 1. Fix m, t in the range of Section 4. Suppose $(v, \mathbf{s})$ attains $f_{m,t}(x)$ and v has positive-dimensional center. On a surface this center is an irreducible curve W through x. The ring $\mathcal{O}_{X,\eta_W}$ is a discrete valuation ring, and v vanishes on its units. Thus $v = a\,\mathrm{ord}_W$ with $a = A(v) > 0$. Choose a very general smooth point y of W as in Lemma 4.5. Even if W is singular at x, we can take regular parameters (u, z) at y with $W = (u = 0)$.

Choose a local frame e for L near y. To compare sections in different degrees j, always expand at the same y, in the same coordinates (u, z), and use e^j as the frame for jL. Locally a nonzero section has the form $s = u^p g$ with $g|_W \neq 0$. Put $\beta = \mathrm{ord}_y(g|_W)$. Comparing first the exponent of u and then that of z gives initial exponent (p, β), and $v(s) = ap$. If Γ_j is the set of all initial exponents of sections in R_j, then

$$\#\Gamma_j = N_j, \qquad \Gamma_j + \Gamma_h \subseteq \Gamma_{j+h}.$$

We want to sum over Γ_m, so we need a basis in which every initial exponent occurs once. Apply Gaussian elimination to a minimizing basis. Pick a basis element s with smallest initial exponent, say (p, β). If another element s' has the same initial exponent, replace s' by $s' - cs$, choosing the scalar c to cancel the coefficient of $u^p z^\beta$. Both sections are divisible by u^p, so $v(s' - cs) \geq ap = v(s')$. The new elements still form a basis. Section values enter $F_{m,t}$ with a minus sign, so this change cannot increase its value. The original basis already attains the minimum. The new basis must attain it too.

Once this term has been eliminated from all the other basis elements, keep s and continue with the remaining elements. Each round fixes one element. At the end we have N_m distinct initial exponents. Since $\#\Gamma_m = N_m$, every $(p, \beta) \in \Gamma_m$ occurs exactly once. From now on $\mathbf{s}$ denotes this basis.

Perturb v along the tangent direction. Keep $v_\varepsilon(u) = a$ and set $v_\varepsilon(z) = \varepsilon > 0$. The center becomes y and $A(v_\varepsilon) = a + \varepsilon$. For this fixed finite basis, all sufficiently small ε give $v_\varepsilon(s) = ap + \varepsilon\beta$. Indeed, increasing the transverse exponent by one increases the old weight by at least a. Choosing ε small enough preserves the initial term for each basis section.

Write $F(\varepsilon) = F_{m,t}(v_\varepsilon, \mathbf{s})$. Since $F(\varepsilon) \geq f_{m,t}(y) = f_{m,t}(x) = F(0)$, the right derivative gives

$$0 \leq (\log F)'(0+) = \frac{1}{t} - T_m, \qquad T_m = \frac{\sum_{(p,\beta)\in\Gamma_m} \beta e^{-ap/m}}{m \sum_{(p,\beta)\in\Gamma_m} e^{-ap/m}}.$$

Per unit increase of the tangential weight, the log discrepancy contributes $1/t$ to the derivative and the weighted vanishing orders contribute a term of size T_m with the opposite sign. We now use the growth of restricted sections to bound this same T_m from below.

Step 2. To bound T_m from below, fix a transverse order p and count the tangential exponents that occur with it. Put $G_p(j) = \{\beta : (p,\beta) \in \Gamma_j\}$ and $r_p(j) = \#G_p(j)$. The group $p = 0$ comes from restriction to the curve. If a section restricts nontrivially to W, its order at y on the curve is β. Eliminating equal initial terms in the image of the restriction map gives

$$r_0(h) = I_W(h) := \dim_{\mathbb{C}} \operatorname{im}\left(R_h \to H^0(W, hL|_W)\right).$$

Here Lemma 5.2 gives constants H_0, K_0 depending only on (X, L) such that $I_W(h) \geq h - K_0$ for every irreducible reduced curve $W \subset X$ and every $h \geq H_0$. The minimizing curve can change with m, so the error bound must work for all degrees and singularities of the curve.

Fix $k > 0$ with kL very ample and embed the surface by $|kL|$. Two general linear forms ℓ_0, ℓ_1 give a finite linear projection $\pi = [\ell_0 : \ell_1] : W \to \mathbb{P}^1$ of degree $D = k(L \cdot W) \geq k$. Write $[T_0 : T_1]$ for homogeneous coordinates on the base $\mathbb{P}^1$ and $K_\eta = \mathbb{C}(\mathbb{P}^1)$ for its function field. The generic fiber is $\operatorname{Spec} \mathbb{C}(W)$. Extending the base field from K_η to its algebraic closure gives the geometric generic fiber. In characteristic zero, $\mathbb{C}(W)/K_\eta$ is a separable extension of degree D, so the geometric generic fiber consists of D distinct points. Choose k of them.

Fix one of these points, say P. For each of the other chosen points, take a linear form vanishing there and nonzero at P. Multiply these $k-1$ forms together and then by ℓ_0, which is nonzero on the generic fiber. The result is a degree-k form that, among our points, is nonzero only at P. Evaluating degree-k forms at these k points therefore gives a map of rank k. Now send each degree-k form F with complex coefficients to $(F/\ell_0^k)|_W \in \mathbb{C}(W)$. After extending scalars, the evaluation argument shows that these images span a space of dimension at least k. Rank is preserved under field extension, so we can choose $F_1, \dots, F_k$ such that F_i/ℓ_0^k are linearly independent over K_η.

For $q \geq k$ there are $q - k + 1$ homogeneous monomials of degree $q - k$ on the base. They are

$$T_0^{q-k-r} T_1^r, \qquad r = 0, 1, \dots, q-k.$$

Pull these monomials back by π. For each F_i, multiply it by every pulled-back monomial. We get $k(q-k+1)$ forms $F_i \ell_0^{q-k-r} \ell_1^r$ of degree q and hence sections in R_{qk}. If their restrictions to W satisfied a complex linear relation, division by ℓ_0^q would give a relation among the F_i/ℓ_0^k over K_η. Each coefficient is a polynomial in the base coordinate T_1/T_0. Linear independence forces every polynomial to vanish, so all the original complex coefficients are zero. This proves

$$I_W(qk) \geq k(q - k + 1).$$

For each residue class modulo k, fix a globally generated $h_r L$ with degree in that class. Multiply by a section of this bundle that does not vanish identically on W. This covers every sufficiently large h and gives the required lower bound $h - K_0$ on the dimension of the restriction image.

To pass from $p = 0$ to any nonempty group, multiply sections. Initial exponents add, so $G_p(j) + G_0(h) \subseteq G_p(j+h)$. Any nonempty finite sets of integers A, B satisfy $\#(A+B) \geq \#A + \#B - 1$. To see why, add the smallest element of B to each element of A in increasing order. Then hold the largest element of A fixed and add the remaining elements of B in increasing order. All these sums are strictly increasing.

Thus, if $r_p(j) > 0$, then for all sufficiently large h,

$$r_p(j + h) \geq r_p(j) + h - K, \qquad K = K_0 + 1.$$

The same K works for every transverse order already present. This count uses the order on the integers. In higher-dimensional tangent spaces the Brunn–Minkowski inequality supplies the corresponding estimate.

Step 3. The elements of a set of r distinct nonnegative integers sum to at least $r(r-1)/2$. So a bound on **how many** tangential exponents occur also bounds **their sum**. Fix the reference degree m and its valuation $a\,\mathrm{ord}_W$. Put

$$w_p = e^{-ap/m}, \qquad Q_j = \sum_p w_p r_p(j), \qquad B_j = \sum_p w_p r_p(j)^2.$$

Use the same weight $w_p = e^{-ap/m}$ at every auxiliary degree j. At degree m, apply the integer sum bound to the tangential exponents in each group. This gives

$$mQ_mT_m = \sum_p w_p \sum_{\beta\in G_p(m)} \beta \geq \frac{1}{2}\sum_p w_p r_p(m)(r_p(m)-1) = \frac{B_m - Q_m}{2}.$$

Thus $T_m + 1/(2m) \geq B_m/(2mQ_m)$. To bound B_m/Q_m from below, first estimate the growth of B_j from degree j to degree $j+h$. Take $h \geq \max\{H_0, K\}$. Fix a transverse order p already present in degree j, so $r_p(j) > 0$. Step 2 gives $r_p(j+h) \geq r_p(j) + (h-K)$. Both sides are nonnegative. Squaring gives

$$r_p(j+h)^2 \geq \big(r_p(j) + h - K\big)^2 = r_p(j)^2 + 2(h-K)r_p(j) + (h-K)^2.$$

Multiply by w_p and drop the nonnegative term $w_p(h-K)^2$. This group then contributes at least $2(h-K)w_p r_p(j)$ more in degree $j+h$ than in degree j. If $r_p(j) = 0$, its contribution is zero in degree j and $w_p r_p(j+h)^2 \geq 0$ in degree $j+h$. Adding the contributions of all transverse orders p therefore gives

$$B_{j+h} \geq \sum_p w_p r_p(j)^2 + 2(h-K)\sum_p w_p r_p(j) = B_j + 2(h-K)Q_j.$$

To iterate up to the reference degree m, fix an integer $\ell \geq 2$ and put $j_i = \lfloor im/\ell \rfloor$ for $1 \leq i \leq \ell$, so $j_\ell = m$. With ℓ fixed and m sufficiently large, every increment $h_i = j_{i+1} - j_i$ is large enough for the estimate. Add the inequalities for consecutive degrees and use $B_{j_1} \geq 0$ to get

$$T_m + \frac{1}{2m} \geq \sum_{i=1}^{\ell-1} \frac{j_{i+1} - j_i - K}{m} \frac{Q_{j_i}}{Q_m}.$$

For the right side to give a useful Riemann sum, we still need a lower bound for Q_j/Q_m.

Fix t and write $f_j = f_{j,t}(x)$. In degree j, test the basis with distinct initial exponents against $(j/m)v$. Its center is still W, and dividing the section value by j gives exactly ap/m. This is why the weight w_p stays the same at each degree. The log discrepancy changes from a to ja/m. By the definition of the infimum and by attainment at degree m,

$$f_j \leq \frac{e^{ja/(mt)}}{N_j} Q_j, \qquad f_m = \frac{e^{a/t}}{N_m} Q_m.$$

Dividing the two expressions gives

$$\frac{Q_j}{Q_m} \geq \frac{N_j}{N_m}\frac{f_j}{f_m} \exp\Big((1 - \frac{j}{m})\frac{a}{t}\Big).$$

The three factors record the dimensions, the infima, and the change of scale. The last, exponential factor will make the eventual lower bound strictly greater than the critical value $1/3$.

Step 4. We can now compare the two bounds for T_m. The lower scale bound $a_* > 0$ from Section 4 is uniform for $t \in [t_0, 3]$ and all sufficiently large degrees. Thus even before choosing t we can define

$$I_* = \int_0^1 s^2 \exp\Big(\frac{(1-s)a_*}{3}\Big)\, ds > \int_0^1 s^2\, ds = \frac{1}{3}.$$

Fix $t \in [t_0, 3)$ close enough to 3 that $I_* > 1/t$. Let $c = \liminf_j f_{j,t}(x) \geq \kappa > 0$ and choose increasing degrees m with $f_{m,t}(x) \to c$. Suppose that for infinitely many of these m there is a minimizing pair with a curve center. Choose one such pair at each of those degrees and apply the first three steps. Both the curve and the valuation may vary with m.

For fixed $s = i/\ell \in (0,1)$ put $j = \lfloor sm \rfloor$. The Hilbert polynomial asymptotics for a surface and the definition of the lower limit give

$$\frac{N_j}{N_m} \to s^2, \qquad \liminf_m \frac{f_j}{f_m} \geq 1.$$

The second estimate uses only $j \to \infty$ and $f_m \to c > 0$. The coefficient a/t is uniformly at least $a_*/3$. Substituting in the ratio estimate of Step 3 gives

$$\liminf_m \frac{Q_{\lfloor sm \rfloor}}{Q_m} \geq s^2 \exp\left((1-s)a_*/3\right).$$

Each interval length divided by m tends to $1/\ell$, and $K/m \to 0$. The sum in Step 3 has finitely many nonnegative terms, so we may take its lower limit term by term as a lower bound. Then let $\ell \to \infty$. We obtain

$$\liminf_m T_m \geq \lim_{\ell \to \infty} \frac{1}{\ell} \sum_{i=1}^{\ell-1} (i/\ell)^2 \exp\left((1 - i/\ell)a_*/3\right) = I_*.$$

Contradiction The tangential perturbation gives $T_m \leq 1/t$, while the growth of restricted sections gives $\liminf_m T_m \geq I_* > 1/t$.

This finishes the choice of degrees. Along the original subsequence with $f_m \to c$, only finitely many degrees can admit a minimizing pair with a curve center. For every remaining degree, **every minimizing pair** has center $\{x\}$.

In Section 6 we will replace one minimizing valuation by another with the same value data. This is why we had to exclude every minimizer with a curve center. It also explains why we needed a uniform positive lower bound for the scale in Section 4. From $a \geq 0$ alone the integral would give only $1/3$, while the target $1/t$ is strictly greater than $1/3$.

Section 6 Constructing a divisor S over x

How does a minimizing valuation give a section nonzero at x?

Fix $t < 3$ and a degree m as in Proposition 5.1. On a common resolution of the zero divisors $D_i = \operatorname{div}(s_i)$ of the chosen basis, take a minimizing valuation v. We want positive coefficients for the divisors $D_i \sim mL$ that give $\Delta \sim_{\mathbb{R}} tL$ with $v(\Delta) = A_X(v)$.

Step 1. First write the conditions on the coefficients as linear inequalities. The variables record how the sections vanish along each divisor. On the fixed model Y, associate to each boundary component E_j the number and vector

$$a_j = A(E_j), \qquad z_j = \left(\operatorname{ord}_{E_j}(D_i)/a_j\right)_{i=1}^{N}.$$

For each open stratum Z whose closure has image containing x, put $C_Z = \operatorname{conv}\{z_j : E_j \supset Z\}$, using only nonempty sets of vertices. On a surface, C_Z is a point or a line segment. With weights normalized by $\sum_j a_j u_j = 1$, the vector of values of a monomial valuation is $\sum_j (a_j u_j) z_j$. Conversely, each such convex combination gives a test of log discrepancy one. Here $N = N_m$.

Write $a = A(v)$ and $q = (v(D_i)/a)_i$. The vector q belongs to some C_Z. No C_Z coming from a stratum with positive-dimensional image contains q. Otherwise the corresponding convex combination would give a valuation that we could rescale to a. It would have exactly the same section values, scale, and objective value, yet its center would contain this positive-dimensional image. That contradicts Proposition 5.1.

We can now find the coefficients from the first derivative at the minimum. Write the relative weights of the exponential terms, the divisor coefficients, and the divisor itself as

$$p_i = \frac{e^{-v(D_i)/m}}{\sum_h e^{-v(D_h)/m}}, \qquad b_i = \frac{t}{m} p_i, \qquad \Delta = \sum_i b_i D_i.$$

Since $p_i > 0$ and $\sum_i p_i = 1$, the divisor Δ is effective and $\Delta \sim_{\mathbb{R}} tL$. Differentiate $\log F_{m,t}(\lambda v, \mathbf{s})$ at $\lambda = 1$. At a minimum the derivative vanishes. Thus $a/t = \sum_i p_i v(D_i)/m$, or equivalently,

$$b \cdot q = 1, \qquad m \sum_i b_i = t < 3.$$

Now keep the scale a fixed. In any polytope C_Z containing q, move along the segment from q toward z. The right derivative satisfies $-(a/t) b \cdot (z - q) \geq 0$. It follows that

$$b \cdot z \leq 1 \qquad (z \in C_Z,\ q \in C_Z).$$

At the vertices z_j of these polytopes this gives $\operatorname{ord}_{E_j}(\Delta) \leq A(E_j)$. At the minimizer, $b \cdot q = 1$ gives $v(\Delta) = A(v)$. On page 15 we use this supporting inequality on **stratum polytopes containing** q to control the components meeting the reduced divisor S.

Step 2. The construction requires a boundary Δ_0 with rational coefficients and a prime divisor E_0 over x such that $\mathrm{ord}_{E_0}(\Delta_0) = A_X(E_0)$. We therefore approximate the pair (q, b) by rational data. We need to retain the strict upper bound on the sum of the coefficients and the full list of polytopes containing q.

In each C_Z containing q, take the smallest face containing q. The linear functional defined by b takes its maximum 1 at q and is therefore constant with value 1 on this face. The intersection of these faces is a rational polytope. In it, choose a rational point q^0 close enough to q to avoid all the finitely many closed polytopes that did not contain q. With q^0 fixed, impose $b^0 \cdot z_j \leq 1$ at the relevant vertices z_j and require $b^0 \cdot q^0 = 1$. The feasible region for these rational linear constraints contains b. On its smallest face containing b, we can approximate b by a rational vector while keeping each coefficient positive and the strict bound on their sum. Lemma 6.1 thus gives

$$\Delta_0 = \sum_i b_i^0 D_i \sim_{\mathbb{Q}} \tau L, \qquad \tau = m \sum_i b_i^0 < 3, \qquad b_i^0 > 0, \quad b^0 \cdot q^0 = 1.$$

The points q^0 and q belong to exactly the same polytopes C_Z. On each polytope containing q^0 we still have $b^0 \cdot z \leq 1$.

To extract E_0, choose a polytope containing q^0 and write q^0 as a rational convex combination of its vertices. The corresponding rational monomial weights span a ray. On a subdivision of the SNC cone, this ray gives the prime divisor E_0, with normalized value vector q^0. Since q^0 avoids the polytopes with positive-dimensional image, E_0 maps to x.

Put $I = \{i : q_i^0 > 0\}$. Choose $M > 0$ so that M/q_i^0 is an integer for every $i \in I$, and define

$$\mathfrak{c} = \sum_{i \in I} \mathcal{O}_X(-(M/q_i^0)D_i).$$

On the current model each $\pi^* D_i$ has a local equation that is a monomial times a unit. Thus $\mathfrak{c}\mathcal{O}_Y$ is a monomial ideal. Choose a suitable toroidal modification and write $\pi : Y \to X$ for the resulting model. We may arrange that Y is smooth and projective, E_0 appears on it, $\mathfrak{c}\mathcal{O}_Y = \mathcal{O}_Y(-G)$, and the relevant components have simple normal crossings.

On a surface, successive blowups of SNC crossing points achieve both goals. Subdivide so that the rational ray appears and the minimum of the monomial weight functions is linear on each cone, then take a smooth subdivision. If we blow up a crossing of F_1 and F_2 with exceptional curve F, we get $a_F = a_1 + a_2$. Likewise, $\mathrm{ord}_F(D_i)$ is the sum of the orders along the two old components. Thus the new vector

$$z_F = \frac{a_1 z_{F_1} + a_2 z_{F_2}}{a_1 + a_2}$$

lies on the old segment. The toroidal modification only subdivides existing cones, so components that meet on the new model can still be studied in a single old polytope.

Step 3. On the modified Y, write $a_F = A(F)$ and $z_F = (\operatorname{ord}_F(D_i)/a_F)_i$ for each boundary component F. Let S be the reduced sum of all components with $x \in \pi(F)$ and $z_F = q^0$. Since $E_0 \subset S$, S is nonempty. By the subdivision above and the choice of q^0 outside the polytopes with positive-dimensional image, every component of S maps to x.

Suppose another boundary component F meets a component E of S. Their crossing corresponds to the segment $[z_E, z_F] = [q^0, z_F]$. It lies in an old polytope containing q^0, so $b^0 \cdot z_F \le 1$. This supporting inequality could be an equality on a whole edge. We want equality only on components whose vector is q^0.

For this, perturb the boundary using the ideal $\mathfrak{c}$ from the previous step. For a rational number $0 < \eta < 1$, to be chosen later, put

$$B = (1-\eta)\pi^*\Delta_0 + \frac{\eta}{M}G, \qquad \mu_F = \min_{i\in I} \frac{z_{F,i}}{q_i^0}.$$

A valuation takes the minimum on a sum of ideals, so $\operatorname{ord}_F(G)/M = \min_{i\in I} \operatorname{ord}_F(D_i)/q_i^0 = a_F\mu_F$. We obtain

$$\frac{\operatorname{coeff}_F B}{a_F} = (1-\eta)b^0 \cdot z_F + \eta\mu_F.$$

If $F \subset S$, both $b^0 \cdot z_F$ and μ_F equal 1. If F meets S, use the positive numbers $b_i^0 q_i^0$ for $i \in I$ as weights. Their sum is 1, and

$$\mu_F \le \sum_{i\in I} b_i^0 q_i^0 \frac{z_{F,i}}{q_i^0} = \sum_{i\in I} b_i^0 z_{F,i} \le b^0 \cdot z_F \le 1.$$

Equality $\operatorname{coeff}_F B = a_F$ requires $\mu_F = b^0 \cdot z_F = 1$. Here a weighted average with positive weights equals its smallest entry, so the ratio for every $i \in I$ must equal 1. Also, since $b_i^0 > 0$, the coordinate $z_{F,i}$ must vanish for every $i \notin I$. Hence $z_F = q^0$. The component F meets S over x, so it belongs to S.

The local configuration in Lemma 6.2 For $F \subset S$, we have $\pi(F) = \{x\}$ and $\operatorname{coeff}_F B = a_F$. Every other component meeting S satisfies $\operatorname{coeff}_F B < a_F$.

Equivalently, near S the boundary $B - K_{Y/X}$ has coefficient 1 along S and coefficient strictly less than 1 along neighboring components. These are the coefficients we will round down. The argument needs no further estimate away from S.

Step 4. To lift from S by Kawamata–Viehweg vanishing, first choose u so that $\mathfrak{c}(uL)$ is globally generated. Then $u\pi^*L - G$ is globally generated and hence nef. From $\Delta_0 \sim_{\mathbb{Q}} \tau L$ we get

$$3\pi^*L - B \equiv (3 - (1-\eta)\tau - \eta u/M)\pi^*L + \frac{\eta}{M}(u\pi^*L - G).$$

At $\eta = 0$ the first coefficient is $3 - \tau > 0$. It stays positive for sufficiently small rational $\eta > 0$. Thus $3\pi^*L - B$ is nef and big. Here we use the strict margin left by choosing $t < 3$ in Sections 4 and 5 and retained in the rational approximation.

Put $D = K_Y + 3\pi^*L - \lfloor B \rfloor$. Since $D - (K_Y + \{B\}) = 3\pi^*L - B$ and the fractional part $\{B\}$ has SNC support, Theorem 2.4 gives $H^1(Y, \mathcal{O}_Y(D)) = 0$. Applying this to the short exact sequence for the reduced Cartier divisor S,

$$0 \longrightarrow \mathcal{O}_Y(D) \longrightarrow \mathcal{O}_Y(D+S) \longrightarrow \mathcal{O}_S(D+S) \longrightarrow 0,$$

we obtain the surjection

$$H^0(Y, D+S) \longrightarrow H^0(S, (D+S)|_S) \longrightarrow 0.$$

To find a section on S to lift, write $D + S = \pi^*P + C$, where $C = K_{Y/X} - \lfloor B \rfloor + S$. Each a_F is an integer, so Lemma 6.2 gives

$$\begin{aligned} &\operatorname{coeff}_F C = (a_F - 1) - a_F + 1 = 0 \qquad (F \subset S), \\ &\operatorname{coeff}_F C = a_F - 1 - \lfloor \operatorname{coeff}_F B \rfloor \geq 0 \quad (F \not\subset S,\ F \cap S \neq \emptyset). \end{aligned}$$

Components outside the chosen boundary have coefficient zero. Hence C is effective on a neighborhood of S and contains no component of S. Since S is reduced and all its components map to x, the map $\pi|_S$ factors scheme-theoretically through x. Thus $\pi^*P|_S = P_x \otimes \mathcal{O}_S$. Take $0 \neq e \in P_x$ and multiply it by the canonical section of C restricted to S. This gives $\sigma_S \in H^0(S, \mathcal{O}_S(D+S))$, nonzero at the generic point of each component of S. Even if S is reducible, these sections agree at intersections. On every component they come from the same fiber vector and the same canonical section on the neighborhood.

By surjectivity, lift σ_S to $\tilde{\sigma} \in H^0(Y, \mathcal{O}_Y(D+S))$. For every nonexceptional curve $F \not\subset S$ we have $\operatorname{coeff}_F C = -\lfloor \operatorname{coeff}_F B \rfloor \leq 0$, so the positive part of C is exceptional. By Lemma 2.5, $\tilde{\sigma}$ descends to X, giving $s \in H^0(X, P)$. If $s(x) = 0$, its pullback would have positive order along every component of S. This contradicts $\operatorname{coeff}_F C = 0$ and the nonvanishing at the generic point.

Conclusion for surfaces We have $s(x) \neq 0$, contradicting the assumption that x is a base point of $|P|$. Since x was arbitrary, $K_X + 3L$ is globally generated.

All dimensions Two estimates that depend on dimension

From surfaces to all dimensions (optional on a first reading)

Let $\dim X = n$ and $P = K_X + (n+1)L$. In Section 4 a full flag still lets us fix a basis. The segment of normalized weights becomes the simplex $\sum_j A(E_j)u_j = 1$. We describe how the mean vanishing order in Section 3 and the tangential estimate in Section 5 change with the dimension.

Step 1. Start with the count in Section 3. The n-dimensional simplex contains $\binom{r+n}{n}$ nonnegative lattice points of total degree at most r. When all these layers are filled, the mean total degree is $nr/(n+1)$. Thus

$$\liminf_j \frac{1}{jN_j} \sum_{\gamma \in H_j} |\gamma|_1 \geq \frac{n}{n+1}(L^n)^{1/n}.$$

If the lower limit equals $n/(n+1)$, then $L^n = 1$. As on page 4, along a subsequence with $\mu_j \to n/(n+1)$ the rescaled set $j^{-1}H_j$ meets every fixed small ball whose closure lies inside the standard simplex. Choose rational $n < t' < n+1$, take points in the slice through $(1/t', \dots, 1/t')$, and clear denominators. This gives sections r_ν of $ht'L$, all of multiplicity nh at x.

First suppose $n > 1$. Blow up x and write $\pi : B \to X$, with exceptional divisor $E \simeq \mathbb{P}^{n-1}$. Remove nhE from $\pi^* r_\nu$ and restrict to E. This gives the lowest-degree homogeneous forms g_ν of the sections r_ν. In local frames the forms generate an ideal $\mathfrak{b}_E$ on E. We want $\operatorname{lct}_z(E, \mathfrak{b}_E) > 1/h$ at every $z \in E$. The strict inequality will make the correction divisor in the lifting argument effective after we round down.

Use homogeneous coordinates $[Z_1 : \cdots : Z_n]$ on E. Let Z^{β_ν} be the initial monomial of g_ν. Our choice of points in the slice ensures that the convex hull of the β_ν/h contains $\mathbf{1} = (1, \dots, 1)$ in its interior relative to the hyperplane $\sum_i u_i = n$. On the chart $Z_k \neq 0$, take $z_i = Z_i/Z_k$ for $i \neq k$. Omit coordinate k from β_ν to obtain γ_ν. The initial monomial on this chart is then z^{γ_ν}. Since the sum determines the missing coordinate, the projected convex hull $\operatorname{conv}\{\gamma_\nu/h\}$ contains $\mathbf{1}$ in its ordinary interior in $\mathbb{R}^{n-1}$.

Given $0 \neq w \in \mathbb{R}^{n-1}_{\geq 0}$, the vector $\mathbf{1}$ is an interior point, so for some sufficiently small $\delta > 0$ the point $\mathbf{1} - \delta w$ is still in the convex hull. The minimum of the linear function $\langle w, - \rangle$ over the hull is at most its value at this point. Thus

$$\frac{1}{h} \min_\nu \langle w, \gamma_\nu \rangle \leq \langle w, \mathbf{1} - \delta w \rangle < \sum_{i \neq k} w_i.$$

Let $\mathfrak{a}_0$ be the ideal generated by the initial monomials. For a prime divisor F over this chart, take $w_i = \operatorname{ord}_F(z_i)$. The left side is then $\operatorname{ord}_F(\mathfrak{a}_0)/h$. The coordinate hyperplanes form a reduced SNC divisor, so $\sum w_i \leq A_E(F)$. If $w = 0$, the ideal has order zero and $A_E(F) > 0$. We therefore have $\operatorname{ord}_F(\mathfrak{a}_0)/h < A_E(F)$ for every F. A log resolution computes the threshold as a minimum over finitely many components. Hence $\operatorname{lct}_z(E, \mathfrak{a}_0) > 1/h$ at every $z \in E$.

Now pass back from the monomials to the forms g_ν. Only finitely many terms occur, so choose integer weights c for which the initial monomial of each form is its unique term of least weight. Put

$$g_{\nu,\lambda}(Z) = \lambda^{-\langle c, \beta_\nu \rangle} g_\nu(\lambda^{c_1} Z_1, \dots, \lambda^{c_n} Z_n).$$

All powers of λ in the expansion are nonnegative. At $\lambda = 0$ only the initial monomial remains, and at $\lambda = 1$ we recover g_ν. Denote the ideal on each fiber by $\mathfrak{a}_\lambda$. By Lemma 2.3 the pairs (z, λ) with $\operatorname{lct}_z(E, \mathfrak{a}_\lambda) \leq 1/h$ form a closed set. Since E is projective, the image of this set on the parameter line is closed and misses 0. Choose a nonzero λ outside the image. All nonzero fibers differ only by a diagonal change of coordinates and nonzero scalar multiples of the generators. Thus $\mathfrak{a}_1 = \mathfrak{b}_E$ also has threshold strictly greater than $1/h$ everywhere on E.

How does this threshold give a section? Let $\mathfrak{b}$ be the base ideal after removing nhE. Take a log resolution $\sigma : T \to B$ of E and $\mathfrak{b}$, write E' for the strict transform of E, and put $\mathfrak{b}\mathcal{O}_T = \mathcal{O}_T(-G)$. Restriction to E' gives $\mathfrak{b}_E\mathcal{O}_{E'} = \mathcal{O}_{E'}(-G|_{E'})$. For every prime divisor F on E', the strict threshold gives

$$\frac{\operatorname{ord}_F(G|_{E'})}{h} < A_E(F) = 1 + \operatorname{ord}_F(K_{E'/E}).$$

The right side is an integer, so the left side rounds down to at most $\operatorname{ord}_F(K_{E'/E})$. Applied to each component, this shows that $C = K_{E'/E} - \lfloor G|_{E'}/h \rfloor$ is effective. By adjunction, the line bundle we get on E' is $P_x \otimes \mathcal{O}_{E'}(C)$. Multiply any $0 \neq e \in P_x$ by the canonical section of C. The result is nonzero at the generic point of E'. Since $n+1-t' > 0$, vanishing gives a surjective restriction map as on page 5, so we can lift this section to T. Relative to $(\pi\sigma)^*P$, the correction divisor has coefficient zero along E' and its positive part is exceptional. Lemma 2.5 lets us descend the section to X with $s(x) \neq 0$.

When $n = 1$, take $B = X$ and $E = x$. After removing hx the sections are nonzero at x, so the restricted ideal is the unit ideal and the same lifting argument applies. Thus in any dimension, if x is a base point of $|P|$, then $\liminf_m \mu_m > n/(n+1)$. Together with $A(\operatorname{ord}_x) = n$, this makes the derivative $n/t - \mu_m$ from Section 4 negative for large m and t sufficiently close to $n+1$.

Step 2. Now consider Section 5. Suppose a minimizing monomial valuation v has center W on X, with $\dim W = d > 0$. On a smooth model Y, the valuation v assigns positive weights $u_1, \dots, u_r$ to the local equations of r crossing divisors. The closure Z of the corresponding stratum has dimension $n-r$, and Z maps onto W. Take a very general point $y \in W$ as in Lemma 4.5. Above y choose a point z where $Z \to W$ is smooth, away from any additional components of the boundary or the relative canonical divisor. The fiber Z_y has dimension $f = n - r - d$ at z.

Choose three groups of local parameters at z. Parameters along Z_y describe motion within the fiber **with y fixed**. Parameters pulled back from W describe motion of the base point along W.

Local parameters	**Weight for v**	**Weight for v_ε**
$\xi_1, \dots, \xi_r$, equations of the crossing divisors	$u_i > 0$	u_i
$\eta_1, \dots, \eta_f$, parameters along the fiber Z_y	0	ε^2
$s_1, \dots, s_d$, parameters pulled back from W	0	ε

Although η_j and s_i vanish at z, neither vanishes identically on Z. At the generic point of Z they are units. Since the original valuation has center Z, it gives them weight zero. On a surface with a curve center W, we have $n = 2$ and $d = r = 1$, hence $f = 0$. This is why page 9 needed just one tangential parameter.

After the perturbation every parameter has positive weight. The center is now z, which maps to y on X. The old weight of a monomial $\xi^\alpha\eta^\gamma s^\beta$ was $\sum_i u_i\alpha_i$. Its new weight and the new log discrepancy are

$$\sum_i u_i\alpha_i + \varepsilon|\beta|_1 + \varepsilon^2|\gamma|_1, \qquad A(v_\varepsilon) = A(v) + d\varepsilon + f\varepsilon^2.$$

Each coordinate divisor receiving a new weight has log discrepancy 1 near z, giving the formula on the right. The fiber parameters contribute only at second order. For each basis section, take the least $|\beta|_1$ among terms attaining the old weight. Their weighted average divided by m is T_m. As on page 9, minimality says that the right derivative of $\log F$ at 0 satisfies $d/t - T_m \geq 0$.

Lemma 5.2 gives $\big(d!I_W(h)\big)^{1/d} \geq h - C_d$ with a constant independent of W. Write $b_p(j) = \big(d!\#G_p(j)\big)^{1/d}$. Thicken the lattice sets by unit cubes and apply Brunn–Minkowski, using the pure tangential initial exponents $\{0,1\}^d$ available in bounded degree. This gives $b_p(j+h) \geq b_p(j) + h - C'_d$ for every group already present. Among sets of fixed volume in the positive orthant, the simplex cut out by a bound on total degree minimizes the first moment of total tangential degree. Thus the surface sum of squares becomes a weighted sum of b_p^{d+1}.

Finally compare the two bounds on T_m. Sum and compare degrees as on pages 11–12, now using $N_{\lfloor sm \rfloor}/N_m \to s^n$. We obtain

$$\liminf_m T_m \geq d \int_0^1 s^n \exp\left((1-s)a_*/(n+1)\right) ds > d/t.$$

For the last strict inequality, first fix the uniform lower bound $a_* > 0$, then choose $t < n+1$ close enough to $n+1$. This rules out every positive-dimensional minimizing center.

Step 3. It remains to lift a section in arbitrary dimension. The normalized value vectors from Section 6 still lie in finitely many rational polytopes. The derivative under rescaling and the supporting inequality still give $b^0 \cdot q^0 = 1$ and $m \sum_i b_i^0 < n+1$. On a surface our toroidal modification automatically satisfies the discrepancy equality below. On a general resolution we must record which components satisfy it.

Let Y be the initial SNC model, with boundary components E_j. Take a higher model $\rho : T \to X$ that retains the prime divisor corresponding to q^0 and principalizes $\mathfrak{c}$. For each component F put

$$a_F = A(F), \qquad r_F = \sum_j \operatorname{ord}_F(E_j) A(E_j) \leq a_F.$$

When $r_F > 0$, the SNC retraction of ord_F on Y has log discrepancy r_F. Write $z' = (\operatorname{ord}_F(D_i)/r_F)_i$. Let S be the reduced divisor formed by the boundary components satisfying $x \in \rho(F)$, $r_F = a_F$, and $z' = q^0$. The rational ray we retained makes S nonempty. The choice of q^0 ensures that every component maps to x.

Suppose F meets S and $r_F > 0$. Its normalized vector z' lies in the same old polytope as q^0, so $b^0 \cdot z' \leq 1$. After perturbation the exact formula is

$$\frac{\operatorname{coeff}_F B}{a_F} = \frac{r_F}{a_F}\Big((1-\eta) b^0 \cdot z' + \eta \min_{i \in I} \frac{z_i'}{q_i^0}\Big).$$

If $r_F = 0$, then all the $\operatorname{ord}_F(D_i)$ and $\operatorname{coeff}_F B$ vanish. Equality with 1 on the right in the displayed formula requires both $r_F = a_F$ and $z' = q^0$. Hence $F \subset S$. We have exactly the equality on S and the strict inequalities on neighboring components that we used when rounding down on a surface.

Replace $3\pi^* L$ by $(n+1)\rho^* L$. Round down and apply Kawamata–Viehweg vanishing as on page 16, then descend the lifted section to X. This gives a section of $K_X + (n+1)L$ nonzero at x. If $n = 1$, a component of S can be the point x itself. Its coefficient in the correction divisor is still zero. This proves the main theorem in every dimension.

To deduce Corollary 6.3 we must also treat $k > n+1$. Even for an ample line bundle L, the space $H^0(X, L)$ can be zero. Set $r = k-n-1$. On the smooth projective variety $Z = X \times \mathbb{P}^r$ take the ample line bundle $M = L \boxtimes \mathcal{O}_{\mathbb{P}^r}(1)$. Since $\dim Z = n+r$, the main theorem applied to (Z, M) shows that its adjoint bundle is globally generated. The product formula gives

$$K_Z + (n+r+1)M = (K_X + kL) \boxtimes \mathcal{O}_{\mathbb{P}^r}(n).$$

Restrict the globally generated bundle on the right to any slice $X \times \{z\}$. This shows that $K_X + kL$ is globally generated. The argument uses the main theorem in all dimensions. The case $r = 0$ is the critical degree itself.

Source OpenAI, *Fujita's freeness conjecture*, September 23, 2026. The theorem, lemma, and equation numbers refer to this version. For surfaces, Section 3 computes multiplicities directly on $\mathbb{P}^1$, Section 5 uses sums of sets of integers, and Section 6 chooses a model by a toroidal modification.

Editor's note These notes grew out of my discussions with ChatGPT as I read the paper. I chose to start with surfaces and treat the dimension-dependent steps separately. ChatGPT drafted the notes from those discussions. I reviewed each section and reread the entire text to check that readers with a background in algebraic geometry could follow the proof from the notes alone.

If you spot an error or find a step hard to follow, I would be glad to hear from you at zhumz@kias.re.kr.