

BOUNDEDNESS OF STABLE MINIMAL MODELS WITH KLT SINGULARITIES

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ABSTRACT. We investigate the singularities and boundedness of a special kind of algebraic varieties so-called stable minimal models, which are constructed and studied by Birkar in [Bir21a, Bir22]. Given a klt stable minimal model with bounded relative volume, if we fix the dimension, Iitaka volume, and a DCC set controlling coefficients, then we show that the singularities of the klt stable minimal model can be controlled uniformly. Furthermore, we prove that with certain bounded data, stable minimal models with klt singularities form a bounded family.

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1. INTRODUCTION

We work over an algebraically closed field k of characteristic zero.

Boundedness properties of algebraic varieties and singularities have been extensively studied in recent years. Regarding algebraic varieties, such as canonically polarized varieties, polarized Calabi-Yau varieties, and Fano varieties, proving boundedness properties is the first step of constructing their potential moduli spaces, see [HMX14, HX15, HMX18, Bir19, Bir21b, Bir21a, Bir22, Bir23a], etc. Moreover, they are also essential tools to prove effective birationalities and effective Iitaka fibrations of algebraic varieties in higher dimensional birational geometry, see [HMX14, BZ16, Bir19, Bir23a, CHL23], etc. As for singularities, Xu and Zhuang proved boundedness of K-semistable log Fano cone singularities recently, and it is a crucial step to prove discreteness of local volumes of klt singularities, see [HLQ23, Zhu23, XZ24, Zhu24], etc.

Since the boundedness results of canonically polarized varieties and polarized Calabi-Yau varieties have been established, it remains to investigate algebraic varieties with intermediate Kodaira dimension. Recent work related to this aspect can be found in [FS20, FHS24, Bir21a, Bir22, Jia22, HH23, Jia23, Fil24, Li24a], etc. In the absence of a natural polarization, unlike canonically polarized varieties, we need to add a polarization. A suitable option is given by Birkar. He defines stable minimal models to be such varieties plus polarization, which generalizes both KSBA-stable varieties and polarized Calabi-Yau varieties. The main purpose of this article is to study boundedness properties of stable minimal models with klt singularities. Here we recall the definition of klt stable minimal models. Note that in general stable minimal models can be defined in slc case.

Definition 1.1. (Klt stable minimal models, [Bir22]) A *klt stable minimal model* (X, B) , A consists of (X, B) and an \mathbb{R} -divisor $A \geq 0$ where

- (X, B) is a projective klt pair,
- $K_X + B$ is semi-ample defining a contraction $f : X \rightarrow Z$, and
- $K_X + B + tA$ is ample for some $t > 0$, i.e. A is ample over Z .

Moreover, if in addition $K_X + B + A$ is ample, then we call (X, B) , A a *klt strongly stable minimal model*.

For the motivation and examples of stable minimal models, we recommend readers to [Bir22, Kol23] and the references therein.

Next we give the definition of families of stable minimal models with certain data.

Definition 1.2. (Families of klt stable minimal models, [Bir22]) Let $d \in \mathbb{N}$, $u, v \in \mathbb{R}^{>0}$, and $\Phi \subset \mathbb{R}^{\geq 0}$.

- (1) A (d, Φ) -*klt stable minimal model* is a klt stable minimal model (X, B) , A such that
 - $\dim X = d$, and
 - $B, A \in \Phi$, which means that the coefficients of B and A are in Φ .

Let $\mathcal{S}_{klt}(d, \Phi)$ denote the set of all (d, Φ) -klt stable minimal models. Denote by $\mathcal{SS}_{klt}(d, \Phi) \subset \mathcal{S}_{klt}(d, \Phi)$ the subset of all klt strongly stable minimal models.

- (2) A $(d, \Phi, \leq u, v)$ -klt stable minimal model is a (d, Φ) -klt stable minimal model $(X, B), A$ such that

- $\text{vol}(A|_F) \leq u$, where F is a general fiber of $f : X \rightarrow Z$, and
- $\text{Ivol}(K_X + B) = v$ (see Definition 2.6 for the definition of Iitaka volumes of \mathbb{R} -divisors).

Let $\mathcal{S}_{klt}(d, \Phi, \leq u, v)$ consist of all $(d, \Phi, \leq u, v)$ -klt stable minimal models. Similarly define $(d, \Phi, \leq u, \leq v)$ -klt stable minimal model and $\mathcal{S}_{klt}(d, \Phi, \leq u, \leq v)$ by replacing the condition “ $\text{Ivol}(K_X + B) = v$ ” with “ $\text{Ivol}(K_X + B) \leq v$ ”.

- (3) When 0 is not an accumulation point of Φ (e.g. when Φ is DCC), we say that a subset $\mathcal{E} \subseteq \mathcal{S}_{klt}(d, \Phi)$ forms a bounded family if there is a positive integer r such that for each $(X, B), A \in \mathcal{E}$, there is a very ample divisor H on X such that $H^d \leq r$ and $(K_X + B + A) \cdot H^{d-1} \leq r$.

Given a klt stable minimal model $(X, B), A \rightarrow Z$ in the family $\mathcal{S}_{klt}(d, \Phi, \leq u, v)$ with fixed d, Φ, u, v , we can prove that the base Z is in a bounded family by Theorem 3.3 and [Bir21a, Theorem 1.4], and the general fiber F is also in a bounded family by [Bir23a, Theorem 6.2]. It is natural to ask whether or not the total space (X, B) belongs to a bounded family.

Conjecture 1.3. *Let $d \in \mathbb{N}$, $u, v \in \mathbb{R}^{>0}$, and $\Phi \subset \mathbb{R}^{\geq 0}$ be a DCC set. If $(X, B), A \in \mathcal{S}_{klt}(d, \Phi, \leq u, v)$, then the set of such (X, B) forms a bounded family.*

Jiao proved that if $u, v \in \mathbb{Q}^{>0}$, and $\Phi \subset \mathbb{Q}^{\geq 0}$, then $\mathcal{S}_{klt}(d, \Phi, u, v)$ is a birationally bounded family (cf. [Jia22, Theorem 1.2]). Note that $\mathcal{S}_{klt}(d, \Phi, u, v)$ is defined similarly to $\mathcal{S}_{klt}(d, \Phi, \leq u, v)$ by replacing the condition “ $\text{vol}(A|_F) \leq u$ ” with “ $\text{vol}(A|_F) = u$ ”. Moreover, in [HH23, Theorem 1.4], Hashizume and Hattori proved the \mathbb{Q} -coefficient version of Conjecture 1.3, in the case that $\text{vol}(A|_F)$ is fixed, and the Iitaka dimension of stable minimal models is one, that is, $(X, B), A \rightarrow Z \in \mathcal{S}_{klt}(d, \Phi, u, v)$ with $\dim Z = 1$.

In [Bir21a, Theorem 1.9], Birkar proved \mathbb{Q} -coefficient version of Conjecture 1.3 for klt strongly stable minimal models $(X, B), A$ with additional conditions that $\text{vol}(A|_F)$ is fixed, and $\text{vol}(K_X + B + A)$ is bounded from above (note that the notation “ $\mathcal{S}_{klt}(d, \Phi, u, v, < w)$ ” in [Bir21a, Theorem 1.9] represents the family of klt strongly stable minimal models defined in [Bir22] and this article). Therefore, we can expect that [Bir21a, Theorem 1.9] also holds for klt stable minimal models. The following conjecture is a weak version of Conjecture 1.3.

Conjecture 1.4. *Let $d \in \mathbb{N}$, $u, v, w \in \mathbb{R}^{>0}$, and $\Phi \subset \mathbb{R}^{\geq 0}$ be a DCC set. Let $(X, B), A \in \mathcal{S}_{klt}(d, \Phi, \leq u, v)$ satisfy that $\text{vol}(K_X + B + A) < w$. Then the set of such $(X, B), A$ forms a bounded family.*

In [Bir22], Birkar proved boundedness of \mathbb{Q} -coefficient slc stable minimal models (X, B) , A of dimension d assuming that the intersection number $(K_X + B)^i \cdot A^{d-i}$ is fixed for any $0 \leq i \leq d$, see [Bir22, Theorem 1.12, Lemma 4.12]. In light of his result, we prove a special case of Conjecture 1.3.

Theorem 1.5. *Let $d \in \mathbb{N}$, $u, v, w \in \mathbb{R}^{>0}$, and $\Phi \subset \mathbb{R}^{\geq 0}$ be a DCC set. Let (X, B) , $A \in \mathcal{S}_{klt}(d, \Phi, \leq u, v)$ satisfy that the intersection number $(K_X + B)^i \cdot A^{d-i} \leq w$ for any $0 \leq i \leq d$. Then the set of such (X, B) , A forms a bounded family.*

Note that the condition on $\text{vol}(A|_F)$ can be removed in Theorem 1.5 (see Remark 5.2).

The main difficulty of the proof comes from \mathbb{R} -coefficients, which is not dealt in [Bir21a, Bir22]. We will explain this more precisely in the sketch of proofs. Indeed, Birkar expected that [Bir22, Theorem 1.12, Theorem 1.14] could be generalized to the real coefficient case (cf. [Bir22, Subsection 11.11]). Theorem 1.5 confirms that \mathbb{R} -coefficient version of [Bir22, Theorem 1.12] holds for klt stable minimal models. However, our method can not be applied to the slc case.

One of the key steps of the proof of Theorem 1.5 is controlling the singularities of klt stable minimal models in $\mathcal{S}_{klt}(d, \Phi, \leq u, v)$ uniformly.

Theorem 1.6. *Let $d \in \mathbb{N}$, $u, v \in \mathbb{R}^{>0}$, and $\Phi \subset \mathbb{R}^{\geq 0}$ be a DCC set. Then there exists $\epsilon \in \mathbb{R}^{>0}$ satisfying the following.*

If (X, B) , $A \in \mathcal{S}_{klt}(d, \Phi, \leq u, v)$, then (X, B) is ϵ -lc.

In light of Theorem 1.6, if we restrict ourselves to ϵ -lc stable minimal models for some fixed $\epsilon \in \mathbb{R}^{>0}$, then in Theorem 1.5 the condition “ $\text{Ivol}(K_X + B) = v$ ” can be replaced with “ $\text{Ivol}(K_X + B) \leq v$ ”. However, in this case, Φ should be a finite set by technical reasons.

Theorem 1.7. *Let $d \in \mathbb{N}$, $\epsilon, u, v, w \in \mathbb{R}^{>0}$, and $\Phi \subset \mathbb{R}^{\geq 0}$ be a finite set. Let (X, B) , $A \in \mathcal{S}_{klt}(d, \Phi, \leq u, \leq v)$ satisfy that (X, B) is ϵ -lc, and the intersection number $(K_X + B)^i \cdot A^{d-i} \leq w$ for any $0 \leq i \leq d$. Then the set of such (X, B) , A forms a bounded family.*

Sketch of proofs. We start with Theorem 1.6. Let (X, B) , $A \in \mathcal{S}_{klt}(d, \Phi, \leq u, v)$. Let $f : X \rightarrow Z$ be a contraction defined by the semi-ample \mathbb{R} -divisor $K_X + B$. First we generalize the effective adjunction formula in [Bir21a, Lemma 7.4] to the \mathbb{R} -coefficient case (see Theorem 3.3). The main new tool of proving Theorem 3.3 is the uniform rational polytope for adjunction formula (cf. [HLX23, Theorem 3.3]).

Next, by Theorem 3.3, there is an adjunction formula $K_X + B \sim_{\mathbb{R}} f^*(K_Z + B_Z + M_Z)$ such that $(Z, B_Z + M_Z) \in \mathcal{F}_{gklt}(\dim Z, \Psi, v)$ for some fixed DCC set $\Psi \subset \mathbb{R}^{\geq 0}$ (see Definition 2.13 for the definition of $\mathcal{F}_{gklt}(\dim Z, \Psi, v)$). Then by [Bir21a, Theorem 1.5], there exists a fixed $\delta \in \mathbb{R}^{>0}$ such that $(Z, B_Z + M_Z)$ is generalized δ -lc.

Take a prime divisor D over X . If D is horizontal over Z , then D determines a prime divisor S over the general fiber F of $f : X \rightarrow Z$. Since (F, B_F) is a klt log Calabi-Yau pair, and $B_F \in \Phi$, then (F, B_F) is τ -lc for some fixed $\tau \in \mathbb{R}^{>0}$ by [Bir19,

Lemma 2.48]. Hence in this case, $a(D, X, B) = a(S, F, B_F) \geq \tau$. On the other hand, if D is vertical over Z , take resolutions $\pi : X' \rightarrow X$ and $\mu : Z' \rightarrow Z$ such that $f' : X' \dashrightarrow Z'$ is a morphism, D is a prime divisor on X' and $E = f'(D)$ is a prime divisor on Z' . Let $K_{X'} + B' = \pi^*(K_X + B)$. By the adjunction formula, $(X', B' + \delta f'^*E)$ is lc over the generic point of E , hence $a(D, X, B) = a(D, X', B') \geq 1 - \delta$. Choose $\epsilon = \min\{\tau, \delta\}$, and we finish the proof.

Next we move on to Theorem 1.5. Let $(X, B), A \in \mathcal{S}_{klt}(d, \Phi, \leq u, v)$ satisfy that the intersection numbers $(K_X + B)^i \cdot A^{d-i} \leq w$. By Theorem 1.6, we can assume that (X, B) is ϵ -lc for some fixed $\epsilon \in \mathbb{R}^{>0}$. As in the proof of [Bir22, Theorem 4.1], the key point of the proof is to find a fixed $\lambda \in \mathbb{R}^{>0}$ such that $(X, B + \lambda A)$ is lc and $K_X + B + \lambda A$ is nef, because then $(X, B + \frac{\lambda}{2}A)$ is $\frac{\epsilon}{2}$ -lc, $K_X + B + \frac{\lambda}{2}A$ is ample, and $\text{vol}(K_X + B + \frac{\lambda}{2}A)$ is bounded from above by the upper bound of the intersection numbers, hence $(X, B), A$ belongs to a bounded family by [HMX14, Theorem 1.3, Theorem 1.6].

In the proof of [Bir22, Theorem 4.1], Birkar showed that there is a fixed $p \in \mathbb{N}$ and an adjunction formula $K_X + B \sim_{\mathbb{R}} f^*(K_Z + B_Z + M_Z)$, such that $L := p(K_Z + B_Z + M_Z)$ is very ample. Then take a general member T of $|L|$, and let S be the pullback of T on X . There is a new stable minimal model $(S, B_S), A_S := A|_S$ and a fibration $S \rightarrow T$. The intersection numbers $(K_S + B_S)^i \cdot A_S^{d-1-i}$ are controlled by $(K_X + B)^i \cdot A^{d-i}$. Therefore, by induction $(S, B_S), A_S$ is bounded. Then using [Bir22, Lemma 4.7], the inversion of adjunction formula, and the length of extremal rays, we can find a fixed $\lambda \in \mathbb{R}^{>0}$ such that $(X, B + \lambda A)$ is lc, and $K_X + B + \lambda A$ is nef.

However, in \mathbb{R} -coefficient case, $K_Z + B_Z + M_Z$ is not a \mathbb{Q} -divisor, hence there is no such $p \in \mathbb{N}$ that makes $p(K_Z + B_Z + M_Z)$ very ample. An alternative way is to decompose $K_Z + B_Z + M_Z$ uniformly, i.e., $K_Z + B_Z + M_Z = \sum_{i=1}^l r_i(K_Z + B_{i,Z} + M_{i,Z})$ such that $r_i \geq 0$, $\sum_{i=1}^l r_i = 1$, and there is a fixed $p \in \mathbb{N}$ such that $L_i := p(K_Z + B_{i,Z} + M_{i,Z})$ is very ample for any i (see Theorem 2.16). But if we take a general member T of $|L_1|$, and denote by $(S, B_S), A_S \rightarrow T$ the induced stable minimal model, then there is no way to control the upper bound of the intersection numbers $(K_S + B_S)^j \cdot A_S^{d-1-j}$ for $0 \leq j \leq d-1$, which means that $\text{vol}(K_S + B_S + A_S)$ cannot be controlled. Hence we cannot use induction in this way.

To solve this problem, our new idea is to consider the image of non-lc locus of $(X, B + tA)$ in Z , which we denote by $P(t)$, i.e., $P(t) = f(\text{Nlc}(X, B + tA))$, and then proving the following two statements inductively:

- (1) There exists a fixed $\lambda_k \in \mathbb{R}^{>0}$ such that $\dim P(\lambda_k) \leq \dim Z - k$.
- (2) There exists a fixed $\mu_k \in \mathbb{R}^{>0}$ such that $K_X + B + \mu_k A$ is nef in dimension $k - 1$ over Z (see Definition 2.17 for the definition of relatively nefness).

Let F be the general fiber of $X \rightarrow Z$, and $(F, B_F), A_F$ be the restriction of $(X, B), A$ on F . Then $(F, B_F), A_F$ is a polarized log Calabi-Yau pair. By [Bir21a, Theorem 6.2], $(F, B_F), A_F$ is bounded. By [Bir21b, Theorem 1.8], there is a fixed $\lambda_1 \in \mathbb{R}^{>0}$ such that $(F, B_F + \lambda_1 A_F)$ is lc. By the inversion of adjunction [Kaw07, Hac14], $\dim P(\lambda_1) \leq \dim Z - 1$. Since $K_X + B$ is semi-ample and A is ample over Z ,

automatically $K_X + B + A$ is nef in dimension 0 over Z . Therefore, the two statements hold for $k = 1$.

When $k \geq 1$, assume there is a fixed $\lambda_k \in \mathbb{R}^{>0}$ such that $\dim P(\lambda_k) \leq \dim Z - k$. Let T be a general complete intersection of dimension k , and S be the pullback of T on X . The adjunction formula gives a pair $(S, B_S + \lambda_k A_S)$ on S . By [FH23, Theorem 1.2, Corollary 1.4], we have $\text{Nlc}(S, B_S + \lambda_k A_S) = S \cap \text{Nlc}(X, B + \lambda_k A)$. Since $\dim P(\lambda_k) \leq \dim Z - k$, we deduce that $\text{Nlc}(S, B_S + \lambda_k A_S)$ is contained in finitely many fibers of $S \rightarrow T$. By the standard method of using the length of extremal rays, there is a fixed $\mu_{k+1} \in \mathbb{R}^{>0}$, such that $K_S + B_S + \mu_{k+1} A_S$ is globally nef. Thus $K_X + B + \mu_{k+1} A$ is nef over Z in dimension k .

On the other hand, assume there is a fixed $\mu_k \in \mathbb{R}^{>0}$ such that $K_X + B + \mu_k A$ is nef in dimension $k - 1$ over Z . First by Theorem 3.3, we construct an adjunction formula $K_X + B \sim_{\mathbb{R}} f^*(K_Z + B_Z + M_Z)$, and decompose $K_Z + B_Z + M_Z$ uniformly, i.e., $K_Z + B_Z + M_Z = \sum_{i=1}^l r_i(K_Z + B_{i,Z} + M_{i,Z})$ such that $r_i \geq 0$, $\sum_{i=1}^l r_i = 1$, and there is a fixed $p \in \mathbb{N}$ such that $L_i := p(K_Z + B_{i,Z} + M_{i,Z})$ is very ample for any i . Then there exists a fixed $m \in \mathbb{N}$ such that $m(K_Z + B_Z + M_Z) - L_1$ is nef. Take T as a general complete intersection of dimension $k - 1$ cut by hypersurfaces in $|L_1|$. Let S be the pullback of T on X . The adjunction formula gives a pair $(S, B_S + \mu_k A_S)$. By the definition of relative nefness in dimension $k - 1$, $K_S + B_S + \mu_k A_S$ is nef. Decreasing μ_k , we can assume that $K_S + B_S + \mu_k A_S$ is ample. Therefore, $(S, B_S), \mu_k A_S$ is a strongly stable minimal model. Although the intersection numbers $(K_S + B_S)^i \cdot A_S^{\dim S - i}$ are still not controlled, we can control $\text{vol}(K_S + B_S + \mu_k A_S)$, because $m(K_Z + B_Z + M_Z) - L_1$ is nef, and $K_X + B + \mu_k A$ is nef in dimension $k - 1$ over Z . Therefore, by some calculations, we prove that $\text{Ivol}(K_S + B_S)$ is in a fixed finite set and $\text{vol}(K_S + B_S + \mu_k A_S)$ is bounded from above. By the lower bound of lc thresholds in strongly stable minimal models (see Theorem 5.1), there is a fixed $\lambda_k \in \mathbb{R}^{>0}$ such that $(S, B_S + \lambda_k A_S)$ is lc. By the inversion of adjunction [Kaw07, Hac14], we conclude that $P(\lambda_k) \cap T = \emptyset$. Note that T is a general complete intersection of dimension $k - 1$, hence $\dim P(\lambda_k) \leq s - k$.

Applying induction on k , we can find such λ that $(X, B + \lambda A)$ is lc outside from finitely many fibers of $f : X \rightarrow Z$ and $K_X + B + \lambda A$ is nef. Applying the standard method of using the length of extremal rays again, decreasing λ , we conclude that $(X, B + \lambda A)$ is globally lc. Therefore, we finish the proof of Theorem 1.5.

At last, we give a brief sketch of the proof of Theorem 1.7. Let $(X, B), A \in \mathcal{S}(d, \Phi, \leq u, \leq v)$ satisfy that the intersection numbers $(K_X + B)^i \cdot A^{d-i} \leq w$. Let $f : X \rightarrow Z$ be the contraction defined by the semi-ample \mathbb{R} -divisor $K_X + B$. By Theorem 3.2, we construct an adjunction formula $K_X + B \sim_{\mathbb{R}} f^*(K_Z + B_Z + M_Z)$ such that $(Z, B_Z + M_Z) \in \mathcal{F}_{gklt}(\dim Z, J, \leq v)$ for some fixed finite set $J \subset \mathbb{R}^{\geq 0}$, and $(Z, B_Z + M_Z)$ is generalized δ -lc for some fixed $\delta \in \mathbb{R}^{>0}$. Then applying the proof of [Fil20, Theorem 1.3], we show that $\text{vol}(K_Z + B_Z + M_Z)$ is in a fixed finite set (see Lemma 5.3). Thus Theorem 1.7 is a consequence of Theorem 1.5.

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2. PRELIMINARIES

2.1. Divisors.

Definition 2.1 (ACC sets and DCC sets). Let $\Phi \subset \mathbb{R}^{\geq 0}$. We say Φ satisfies the *ascending chain condition* (ACC) if it does not contain an infinite strictly increasing sequence. We say Φ satisfies the *descending chain condition* (DCC) if it does not contain an infinite strictly decreasing sequence.

Definition 2.2 (Coefficients of divisors). Let $\delta \in \mathbb{R}^{>0}$, $\Phi \subset \mathbb{R}^{\geq 0}$, and D be an \mathbb{R} -divisor on a normal variety X . Write $D = \sum a_i D_i$ where D_i 's are different Weil divisors on X . We denote $D \in \Phi$ if $a_i \in \Phi$ for any i , and denote $D \geq \delta$ (resp. $D \leq \delta$) if $a_i \geq \delta$ (resp. $a_i \leq \delta$) for any i .

Definition 2.3 (Contractions). We say a projective morphism $f : X \rightarrow Z$ between normal varieties is a *contraction* if $f_* \mathcal{O}_X = \mathcal{O}_Z$. In particular, f has connected fibers.

Definition 2.4 (Horizontal part and vertical part of divisors). Let $f : X \rightarrow Z$ be a contraction between normal varieties. Let D be an \mathbb{R} -divisor on X . We say that D is *vertical* over Z if $f(\text{Supp } D)$ is a proper subset of Z . We say that D is *horizontal* over Z if the induced map $\text{Supp } D \rightarrow Z$ is dominant.

Given an \mathbb{R} -divisor D on X , there is a unique decomposition $D = D^h + D^v$ such that

- $\text{Supp } D^h, \text{Supp } D^v$ have no common components,
- every component of $\text{Supp } D^h$ is horizontal over Z , and
- D^v is vertical over Z .

We call D^h the *horizontal part* of D and D^v the *vertical part* of D with respect to $f : X \rightarrow Z$.

Definition 2.5 (Invariant Iitaka dimensions, [Cho08, Definition 2.2.1]). Let D be an \mathbb{R} -divisor on a projective normal variety X . We define the *invariant Iitaka dimension* $\kappa_l(X, D)$ as follows. If $|D|_{\mathbb{R}} \neq \emptyset$, let $\kappa_l(X, D) = \kappa(X, D')$ for some \mathbb{R} -divisor $D' \in |D|_{\mathbb{R}}$. Here, the right hand side is the usual Iitaka dimension of D' . Note that in this case $\kappa_l(X, D)$ does not depend on the choice of D' by [Cho08, Corollary 2.1.4]. If $|N|_{\mathbb{R}} = \emptyset$, let $\kappa_l(X, D) = -\infty$.

Next we generalize the definition of Iitaka volume in [Li24a] to \mathbb{R} -divisors. It is also called pseudo-volume in [Cho08].

Definition 2.6. (Iitaka volumes of \mathbb{R} -divisors, [Li24a, Definition 1.1], see also [Cho08, Definition 2.1.1]) Let D be an \mathbb{R} -divisor on a projective normal variety X with invariant Iitaka dimension $\kappa_l(D)$. We define *Iitaka volume* $\text{Ivol}(D)$ of D as follows. If

$\kappa_\iota(D) \geq 0$, let D' be an element of $|D|_{\mathbb{R}}$, and then

$$\text{Ivol}(D) := \limsup_{m \rightarrow \infty} \frac{h^0(\lfloor mD' \rfloor)}{m^{\kappa_\iota(D)}/\kappa_\iota(D)!}.$$

Note that in this case $\text{Ivol}(D)$ does not depend on the choice of D' by [Cho08, Corollary 2.1.4]. If $\kappa_\iota(D) = -\infty$, then let $\text{Ivol}(D) = 0$.

If $f : X \rightarrow Z$ is a contraction between two normal varieties and $D \sim_{\mathbb{R}} f^*L$ for some big \mathbb{R} -divisor L on Z , then $\text{Ivol}(D) = \text{vol}(L)$.

Definition 2.7 (b-divisors). Let X be a normal variety. A *b-divisor* \mathbf{M} is a collection of \mathbb{R} -divisors M_Y on Y for each birational contraction $Y \rightarrow X$ from a normal variety and satisfies the following: if $Y' \rightarrow Y \rightarrow X$ are birational contractions, then the pushdown of $M_{Y'}$ on Y is M_Y .

We say a b-divisor \mathbf{M} is *b- \mathbb{R} -Cartier* if there is a birational contraction $Y \rightarrow X$ such that

- M_Y is \mathbb{R} -Cartier, and
- if $Y' \rightarrow Y$ is a birational contraction, then $M_{Y'}$ is the pullback of M_Y .

In this case, we say that the b- \mathbb{R} -Cartier divisor \mathbf{M} descends on Y and is represented by M_Y . Note that the representation is not unique, if $Y' \rightarrow X$ is another birational contraction and $M_{Y'}$ is an \mathbb{R} -Cartier divisor on Y' , then M_Y and $M_{Y'}$ define the same b- \mathbb{R} -Cartier b-divisor if the pullbacks of M_Y and $M_{Y'}$ to a common resolution of Y and Y' are the same.

2.2. (Generalized) Pairs and Singularities.

Definition 2.8 (Pairs and Singularities). Let X be a normal quasi-projective variety and B be an \mathbb{R} -divisor on X . We say that (X, B) is a *sub-pair* if $K_X + B$ is \mathbb{R} -Cartier. If in addition $B \geq 0$, then (X, B) is a *pair*.

Let D be a prime divisor over X , i.e. there is a birational model over X such that D is a prime divisor on this model. Let $W \rightarrow X$ be a log resolution of a sub-pair (X, B) so that D is a prime divisor on W . Let $K_W + B_W$ be the pullback of $K_X + B$. Define the *log discrepancy* of the prime divisor D as $1 - \mu_D B_W$, where $\mu_D B_W$ means the coefficient of D in B_W . We denote the log discrepancy of D with respect to (X, B) as $a(D, X, B)$.

We say that a sub-pair (X, B) is *sub-klt* (resp. *sub-lc*, *sub- ϵ -lc*) if $a(D, X, B) > 0$ (resp. $a(D, X, B) \geq 0$, $a(D, X, B) \geq \epsilon$) for every prime divisor D over X . If (X, B) is a pair, then we remove the sub and say the pair is klt (resp. lc, ϵ -lc).

Let (X, B) be a sub-pair. A *non-klt place* (resp. *non-lc place*) is a prime divisor D over X such that $a(D, X, B) \leq 0$ (resp. $a(D, X, B) < 0$). A *non-klt center* (resp. *non-lc center*) is the image of a non-klt place (resp. non-lc place). The *non-klt locus* (resp. *non-lc locus*) of (X, B) is the union of all non-klt places (resp. non-lc places) of (X, B) and denoted as $\text{Nklt}(X, B)$ (resp. $\text{Nlc}(X, B)$).

Definition 2.9 (Generalized pairs and Singularities, [BZ16, Definition 1.4, Definition 4.1]). A *generalized sub-pair* consists of

- a normal variety X equipped with a projective morphism $X \rightarrow Z$,

- an \mathbb{R} -divisor B on X , and
- a b- \mathbb{R} -Cartier b-divisor over X , represented by a projective birational morphism $f : X' \rightarrow X$ and an \mathbb{R} -Cartier \mathbb{R} -divisor M' on X'

such that M' is nef over Z and $K_X + B + M$ is \mathbb{R} -Cartier, where $M := f_*M'$. If in addition $B \geq 0$, then $(X, B + M)$ is a *generalized pair*. Since a b- \mathbb{R} -Cartier b-divisor is defined birationally, in practice we will often replace X' with a higher model and replace M' with its pullback. In this article, we omit Z but say the generalized pair is projective when Z is a point.

Let D be a prime divisor over X . Replace X' with a log resolution of (X, B) such that D is a prime divisor on X' . We can write

$$K_{X'} + B' + M' = \pi^*(K_X + B + M).$$

Then we define the *generalized log discrepancy* of D to be $a(D, X, B + M) = 1 - \mu_D B'$.

We say that $(X, B + M)$ is *generalized klt* (resp. *generalized lc*, *generalized ϵ -lc*) if $a(D, X, B + M) > 0$ (resp. $a(D, X, B + M) \geq 0$, $a(D, X, B + M) \geq \epsilon$) for every prime divisor D over X .

2.3. Adjunction formulas for fiber spaces. We recall the construction of adjunction formulas for fiber spaces based on [Kaw98, Amb99, Amb05]. Let (X, B) be a projective sub-pair and let $f : X \rightarrow Z$ be a contraction between quasi-projective normal varieties with $\dim Z > 0$ such that (X, B) is sub-lc near the generic fiber of f and $K_X + B \sim_{\mathbb{R}} 0/Z$.

Fix a prime divisor D on Z and let t_D be the lc threshold of f^*D with respect to (X, B) over the generic point of D , i.e. t_D is the largest number so that $(X, B + t_D f^*D)$ is sub-lc over the generic point of D . Now let $b_D = 1 - t_D$ and by basic argument there are finitely many prime divisors D' on Z such that $b_{D'} \neq 0$. Hence we can define $B_Z = \sum b_D D$, where the sum runs over all the prime divisors on Z .

Since $K_X + B \sim_{\mathbb{R}} 0/Z$, there is an \mathbb{R} -Cartier \mathbb{R} -divisor L_Z on Z such that $K_X + B \sim_{\mathbb{R}} f^*L_Z$. Let $M_Z = L_Z - (K_Z + B_Z)$ and we have the following *adjunction formula*

$$K_X + B \sim_{\mathbb{R}} f^*(K_Z + B_Z + M_Z).$$

We call B_Z the *discriminant divisor* and M_Z the *moduli divisor* of (X, B) with respect to $f : X \rightarrow Z$. Note that B_Z is uniquely determined but M_Z is determined only up to \mathbb{R} -linear equivalence.

Take a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{\pi} & X \\ f' \downarrow & & \downarrow f \\ Z' & \xrightarrow{\mu} & Z \end{array}$$

such that μ and π are birational contractions. Let $K_{X'} + B'$ be the pullback of $K_X + B$ on X' and similarly we can define a discriminant divisor $B_{Z'}$ and $L_{Z'} = \mu^*L_Z$ gives a moduli divisor $M_{Z'}$ so that

$$K_{X'} + B' \sim_{\mathbb{R}} f'^*(K_{Z'} + B_{Z'} + M_{Z'}).$$

It is easy to see that B_Z is the pushdown of $B_{Z'}$ and M_Z is the pushdown of $M_{Z'}$. Therefore, B_Z and M_Z can be regarded as b-divisors.

The following lemma shows that when (X, B) is lc over the generic point of Z , $(Z, B_Z + M_Z)$ is a generalized pair.

Lemma 2.10. (Adjunction formula for \mathbb{R} -coefficients, [JLX22, Theorem 2.23]) *With the above notation and assumptions, suppose that (X, B) is lc over the generic point of Z . Then $M_{Z'}$ is nef on some high resolution $Z' \rightarrow Z$, and $(X, B + M)$ is a generalized pair.*

Here we recall the M^cKernan-Shokurov type conjecture which was proved by Birkar recently.

Lemma 2.11. ([Bir23b, Theorem 1.8]) *Let $d \in \mathbb{N}$, and $u, v, \epsilon \in \mathbb{R}^{>0}$. Then there exists $\delta \in \mathbb{R}^{>0}$ depending only on d, u, v, ϵ satisfying the following.*

Assume that

- (X, B) is an ϵ -lc pair,
- $f : X \rightarrow Z$ is a contraction and $\dim X - \dim Z \leq d$,
- $K_X + B \sim_{\mathbb{R}} 0/Z$,
- A is an effective \mathbb{R} -divisor on X such that $A \geq u$, and
- $0 < \text{vol}(A|_F) < v$ for the general fibers F of f .

Then the generalized pair $(Z, B_Z + M_Z)$ given by the adjunction formula

$$K_X + B \sim_{\mathbb{R}} f^*(K_Z + B_Z + M_Z)$$

is generalized δ -lc.

Proof. Since $A \geq u$, we conclude that

$$0 < \text{vol}(A_{\text{red}}|_F) \leq \text{vol}\left(\frac{1}{u}A|_F\right) \leq \frac{v}{u^{\dim F}}$$

where F is a general fiber of f .

Using approximation, we can write $B = \sum_{i=1}^l r_i B_i$ such that

- $\sum_{i=1}^l r_i = 1$ and $r_i \geq 0$ for any i ,
- r_i 's are \mathbb{Q} -linear independent, and
- (X, B_i) is an $\frac{\epsilon}{2}$ -lc \mathbb{Q} -pair for any i .

Moreover, we have $K_X + B_i \sim_{\mathbb{Q}} 0/Z$ by [HLS24, Lemma 5.3].

Let D be a prime divisor over Z . Take a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{\pi} & X \\ f' \downarrow & & \downarrow f \\ Z' & \xrightarrow{\mu} & Z \end{array}$$

such that μ and π are birational contractions and D is a prime divisor on Z' . Write $K_{X'} + B' = \pi^*(K_X + B)$ and $K_{X'} + B'_i = \pi^*(K_X + B_i)$. Let t_D (resp. $t_{i,D}$) be the lc threshold of f'^*D with respect to (X', B') (resp. (X', B'_i)) over the generic point η_D of D . Since $(X', B'_i + t_{i,D}f'^*D)$ is sub-lc over η_D , $(X', B' + \sum_{i=1}^l r_i t_{i,D}f'^*D)$ is also sub-lc over η_D , hence $t_D \geq \sum_{i=1}^l r_i t_{i,D}$.

Applying [Bir23b, Theorem 1.8] to $(X, B_i), A_{\text{red}}$, we deduce that there exists $\delta \in \mathbb{R}^{>0}$ depending only on d, u, v, ϵ such that $t_{i,D} \geq \delta$. Therefore,

$$t_D \geq \sum_{i=1}^l r_i t_{i,D} \geq \sum_{i=1}^l r_i \delta = \delta.$$

Hence the discriminant b-divisor \mathbf{B}_Z has coefficients $\leq 1 - \delta$ and $(Z, B_Z + M_Z)$ is generalized δ -lc. \square

2.4. Bounded families.

Definition 2.12 (Bounded families of couples and pairs). A *couple* consists of a projective normal variety X and a reduced divisor D on X . We say that two couples (X, D) and (X', D') are isomorphic if there is an isomorphism $X \rightarrow X'$ mapping D onto D' .

Let \mathcal{P} be a set of couples. Assume that

- there exist finitely many projective morphisms $V^i \rightarrow T^i$ of varieties,
- C^i is a reduced divisor on V^i , and
- for each $(X, D) \in \mathcal{P}$ there exists an i , a closed point $t \in T^i$ and an isomorphism $\phi : V_t^i \rightarrow X$ such that (V_t^i, C_t^i) is a couple and $\phi_* C_t^i \geq D$.

Then we say that \mathcal{P} is *bounded*. This is equivalent to say that there is a positive integer r such that for each $(X, D) \in \mathcal{P}$, we can find a very ample divisor A on X such that $A^{\dim X} \leq r$ and $D \cdot A^{\dim X - 1} \leq r$ (cf. [Bir19, Lemma 2.20]).

A set of projective pairs (X, B) is said to be bounded if the set of $(X, \text{Supp } B)$ forms a bounded family of couples.

2.5. Families of generalized pairs.

Definition 2.13 ([Bir21a, Definition 1.1]). Let $d \in \mathbb{N}$, $\Phi \subset \mathbb{R}^{\geq 0}$, and $v \in \mathbb{R}^{>0}$.

- (1) Let $\mathcal{F}_{gklt}(d, \Phi)$ be the set of projective generalized pairs $(X, B + M)$ with data $X' \rightarrow X$ and M' such that
 - $(X, B + M)$ is generalized klt of dimension d ,
 - $B \in \Phi$,
 - $M' = \sum \mu_i M'_i$ where M'_i is Cartier nef and $\mu_i \in \Phi$ for any i , and
 - $K_X + B + M$ is ample.

- (2) Let

$$\mathcal{F}_{gklt}(d, \Phi, v) \subseteq \mathcal{F}_{gklt}(d, \Phi)$$

consist of those $(X, B + M)$ such that $\text{vol}(K_X + B + M) = v$. Similarly, let

$$\mathcal{F}_{gklt}(d, \Phi, \leq v) \subseteq \mathcal{F}_{gklt}(d, \Phi)$$

consist of those $(X, B + M)$ such that $\text{vol}(K_X + B + M) \leq v$.

Here we give a lemma to show that if $(X, B + M) \in \mathcal{F}_{gklt}(d, \Phi, v)$, then we can control the Cartier index of any \mathbb{Q} -Cartier Weil divisor on X .

Lemma 2.14. *Let $d \in \mathbb{N}$, $v \in \mathbb{R}^{>0}$, and $\Phi \subset \mathbb{R}^{\geq 0}$ be a DCC set. Then there exists $N \in \mathbb{N}$ depending only on d, Φ, v such that for any $(X, B + M) \in \mathcal{F}_{gklt}(d, \Phi, v)$ and any \mathbb{Q} -Cartier Weil divisor D on X , the Cartier index of D divides N .*

Proof. Let $(X, B + M) \in \mathcal{F}_{gklt}(d, \Phi, v)$ and D be a \mathbb{Q} -Cartier Weil divisor on X . By Step 7 of the proof of [Bir21a, Theorem 1.4], there is a boundary Θ on X such that (X, Θ) is ϵ -lc for some positive real number ϵ depending only on d, Φ, v , and (X, Θ) belongs to a log bounded family \mathcal{P} . By [HLQ23, Theorem 1.10], There is a positive integer N depending only on ϵ, \mathcal{P} , hence depending only on d, Φ, v , such that the Cartier index of D divides N . \square

2.6. Decomposition of \mathbb{R} -coefficient generalized pairs in $\mathcal{F}_{gklt}(d, \Phi, v)$. In this subsection, we decompose an \mathbb{R} -coefficient generalized pair $(X, B + M) \in \mathcal{F}_{gklt}(d, \Phi, v)$ into \mathbb{Q} -coefficient generalized pairs $(X, B_i + M_i)$ such that a bounded multiple of $K_X + B_i + M_i$ is very ample.

Lemma 2.15. *Let $d \in \mathbb{N}$. Then there exists $m \in \mathbb{N}$ depending only on d satisfying the following. Assume that*

- $(X, B + M)$ is a generalized klt pair of dimension d ,
- L is an ample Cartier divisor on X , and
- $L - K_X - B - M$ is nef and big.

Then mL is very ample.

Proof. By the proof of [Xie22, Lemma 2.4], there exists an effective \mathbb{R} -divisor Δ such that (X, Δ) is klt and $L - K_X - \Delta$ is ample. Then by effective base point free theorem [Kol93, Theorem 1.1], $|mL|$ is base point free for some positive integer m depending only on d . By effective very ampleness lemma [Fuj17, Lemma 7.1], replacing m with a bounded multiple, we conclude that mL is very ample. \square

Theorem 2.16. *Let $d \in \mathbb{N}$, $v \in \mathbb{R}^{>0}$, and $I \subset \mathbb{R}^{\geq 0}$ be a finite set. Then there exists a finite set $J \subset \mathbb{R}^{>0}$ and $p \in \mathbb{N}$ depending only on d, I, v satisfying the following.*

If $(X, B + M) \in \mathcal{F}_{gklt}(d, I, v)$, then we can decompose $K_X + B + M$ as follows:

$$K_X + B + M = \sum_{i=1}^l r_i (K_X + B_i + M_i)$$

such that

- $\sum_{i=1}^l r_i = 1$ and $r_i \in J$ for any i ,
- $(X, B_i + M_i)$ is a generalized klt pair with nef part M'_i on some high resolution $X' \rightarrow X$ for any i , and
- $p(K_X + B_i + M_i)$ is very ample and pM'_i is Cartier nef for any i .

Proof. Pick a generalized pair $(X, B + M) \in \mathcal{F}_{gklt}(d, I, v)$ with data $X' \rightarrow X$ and M' . By [Che23, Theorem 3.15] we can write

$$K_X + B + M = \sum_{i=1}^l r_i (K_X + B_i + M_i)$$

such that

- $\sum_{i=1}^l r_i = 1$ and $r_i \in J$, where $J \subset \mathbb{R}^{>0}$ is a finite set depending only on d, I ,
- $(X, B_i + M_i)$ is a generalized klt pair with nef part M'_i on X' for any i ,
- $M' = \sum_{i=1}^l r_i M'_i$, and

- there exists a positive integer p depending only on d, I such that $p(K_X + B_i + M_i)$ is integral, and pM_i is Cartier nef for any i .

Note that $l \leq N := \frac{1}{\min J}$, because $\sum_{i=1}^l r_i = 1$. By Lemma 2.14, replacing p with a bounded multiple we can assume that $p(K_X + B_i + M_i)$ is Cartier. Now p depends on d, I, v .

Consider the set

$$\Lambda := \left\{ \sum_{i=1}^h \frac{\nu_i n_i}{p} \mid n_i \geq -2dp, n_i \in \mathbb{Z}, \nu_i \in J, h \leq N \right\}.$$

It is easy to see that Λ is a DCC set, hence we can take $\delta := \min\{\alpha > 0 \mid \alpha \in \Lambda\}$.

If $K_X + B_i + M_i$ is not nef for some i , let R be a $(K_X + B_i + M_i)$ -negative extremal ray. Let C be an extremal curve of R , which means that there exists an ample divisor H such that

$$H \cdot C = \min\{H \cdot \Gamma \mid [\Gamma] \in R\}.$$

By the length of extremal ray for generalized pairs [HL22, Proposition 3.13, Lemma 3.5],

$$(K_X + B_i + M_i) \cdot C \geq -2d$$

for $1 \leq i \leq l$. Since $p(K_X + B_i + M_i)$ is Cartier, we deduce that $p(K_X + B_i + M_i) \cdot C$ is an integer in $[-2dp, \infty)$. Since $K_X + B + M$ is ample and

$$\begin{aligned} (K_X + B + M) \cdot C &= \sum_{i=1}^l r_i (K_X + B_i + M_i) \cdot C \\ &= \sum_{i=1}^l \frac{r_i}{p} (p(K_X + B_i + M_i) \cdot C) \in \Lambda, \end{aligned}$$

hence $(K_X + B + M) \cdot C \geq \delta$. Therefore,

$$\left(\frac{\delta}{2d + \delta} (K_X + B_i + M_i) + \frac{2d}{2d + \delta} (K_X + B + M) \right) \cdot C \geq 0.$$

This argument implies that

$$\frac{\delta}{2d + \delta} (K_X + B_i + M_i) + \frac{2d}{2d + \delta} (K_X + B + M)$$

is nef. Now we take

$$K_X + \tilde{B}_i + \tilde{M}_i := \frac{4d + \delta}{4d + 2\delta} (K_X + B + M) + \frac{\delta}{4d + 2\delta} (K_X + B_i + M_i),$$

then

$$K_X + B + M = \sum_{i=1}^l r_i (K_X + \tilde{B}_i + \tilde{M}_i).$$

Since $K_X + B + M$ is ample, we conclude that $K_X + \tilde{B}_i + \tilde{M}_i$ is also ample. Consider the convex hull \mathcal{H} spanned by $K_X + \tilde{B}_i + \tilde{M}_i$, i.e.

$$\mathcal{H} := \left\{ \sum_{i=1}^l \lambda_i (K_X + \tilde{B}_i + \tilde{M}_i) \mid \lambda_i \geq 0, \sum_{i=1}^l \lambda_i = 1 \right\}.$$

Since $K_X + B + M$ is an interior point of \mathcal{H} , we can choose $K_X + \overline{B}_i + \overline{M}_i \in \mathcal{H}$ such that

- $K_X + B + M = \sum_{i=1}^l \overline{r}_i (K_X + \overline{B}_i + \overline{M}_i)$, where \overline{r}_i belongs to a finite set $\overline{J} \subset \mathbb{R}^{>0}$ depending only on J, d, δ , hence depending only on d, I, v ,
- $(X, \overline{B}_i + \overline{M}_i)$ is a generalized klt pair with nef part \overline{M}'_i on X' for each i ,
- $(K_X + \overline{B}_i + \overline{M}_i)$ is ample for each i , and
- there exists a positive integer \overline{p} depending only on J, d, δ, p , hence depending only on d, I, v such that $\overline{p}(K_X + \overline{B}_i + \overline{M}_i)$ is integral and $\overline{p}\overline{M}'_i$ is Cartier nef for any i .

By Lemma 2.14, replacing \overline{p} with a bounded multiple we can assume that $\overline{p}(K_X + \overline{B}_i + \overline{M}_i)$ is Cartier. By Lemma 2.15, replacing \overline{p} again with a bounded multiple we can assume that $\overline{p}(K_X + \overline{B}_i + \overline{M}_i)$ is very ample. Now replace r_i with \overline{r}_i , J with \overline{J} , $(X, B_i + M_i)$ with $(X, \overline{B}_i + \overline{M}_i)$, and p with \overline{p} , and then we finish the proof. \square

2.7. Relative nefness.

Definition 2.17. Let s, k be integers such that $0 \leq k \leq s$. Let $f : X \rightarrow Z$ be a contraction between projective normal varieties with $\dim Z = s$. Let (X, B) be a pair. We say $K_X + B$ is *nef in dimension k over Z* , if for any very ample divisors H_1, \dots, H_{s-k} on Z , the following is satisfied.

Let L_i be a general member of $|H_i|$ and $R_i = f^{-1}L_i$. Take $S = \cap_{i=1}^{s-k} R_i$ and write

$$K_S + B_S = (K_X + B + \sum_{i=1}^{s-k} R_i)|_S,$$

then $K_S + B_S$ is nef.

Note that if $K_X + B$ is nef over Z , then $K_X + B$ is nef in dimension 0 over Z .

The following lemma will be used in the proof of Theorem 1.5.

Lemma 2.18. *Let d, s, k be integers such that $0 \leq k \leq s \leq d$. Assume that*

- $f : X \rightarrow Z$ is a contraction between projective normal varieties with $\dim X = d$ and $\dim Z = s$,
- (X, B) is a pair and $K_X + B$ is nef in dimension k over Z ,
- H_1, \dots, H_{s-k} are very ample divisors on Z , and
- N is a nef \mathbb{R} -divisor on X .

Then

$$(K_X + B + N + \sum_{i=1}^{s-k} f^* H_i)^{d-s+k} \cdot f^* H_1 \cdot \dots \cdot f^* H_{s-k} \geq 0.$$

Proof. By definition, if we take L_i as a general member of $|H_i|$, $R_i = f^{-1}L_i$ and $S = \cap_{i=1}^{s-k} R_i$, then

$$K_S + B_S = (K_X + B + \sum_{i=1}^{s-k} R_i)|_S$$

is nef, hence $K_S + B_S + N_S$ is nef, where $N_S = N|_S$. Therefore,

$$\begin{aligned}
& (K_X + B + N + \sum_{i=1}^{s-k} f^* H_i)^{d-s+k} \cdot f^* H_1 \cdots f^* H_{s-k} \\
&= (K_X + B + N + \sum_{i=1}^{s-k} f^* L_i)^{d-s+k} \cdot f^* L_1 \cdots f^* L_{s-k} \\
&= (K_X + B + N + \sum_{i=1}^{s-k} R_i)^{d-s+k} \cdot R_1 \cdots R_{s-k} \\
&= (K_S + B_S + N_S)^{d-s+k} \geq 0.
\end{aligned}$$

□

3. EFFECTIVE ADJUNCTION FORMULA WITH REAL COEFFICIENTS

In this section we extend the effective adjunction formula [Bir21a, Lemma 7.4] to the real coefficients case. The main new tool is uniform rational polytopes for canonical bundle formulas developed in [HLX23]. The effective adjunction formula is one of the main ingredients in the proof of Theorem 1.5 and Theorem 1.6.

3.1. Cartier index of moduli divisors. Given $q \in \mathbb{N}$ and two \mathbb{R} -divisors C, D on a normal variety X , if $qC \sim qD$, then we write $C \sim_q D$.

Lemma 3.1. *Let $d, q \in \mathbb{N}$, $u \in \mathbb{R}^{>0}$ and $\Phi \subset \mathbb{R}^{\geq 0}$ be a DCC set. Then there exists $p \in \mathbb{N}$ depending only on d, q, u, Φ satisfying the following. Assume that*

- (X, B) is a projective lc pair of dimension d ,
- $f : X \rightarrow Z$ is a contraction with $K_X + B \sim_{\mathbb{Q}} 0/Z$,
- we have an adjunction formula

$$K_X + B \sim_q f^*(K_Z + B_Z + M_Z),$$

- $A \in \Phi$ is an effective \mathbb{R} -divisor on X ,
- over some non-empty open subset $U \subseteq Z$: $(X, B + tA)$ is lc for some $t > 0$, and A is relatively semi-ample, and
- for the general fiber F of f , we have $0 < \text{vol}(A|_F) \leq u$.

Then $pM_{Z'}$ is Cartier on some high resolution $Z' \rightarrow Z$.

Proof. We follow the proof of [Bir21a, Lemma 7.4].

Step 1. Take a resolution $Z' \rightarrow Z$ so that \mathbf{M} descends to Z' , and a log resolution $W \rightarrow X$ of (X, B) such that $W \dashrightarrow Z'$ is a morphism. Let Δ_W be the sum of the horizontal/ Z' part of reduced exceptional divisors and the birational transform of the horizontal/ Z part B^h of B . After finitely many blow ups and changing Δ_W accordingly, we can assume that every non-klt center of (W, Δ_W) is horizontal over Z' .

Run an MMP on $K_W + \Delta_W$ over X with scaling of some ample divisor. Since over the generic point η_Z of Z , $K_W + \Delta_W$ is the sum of the pullback of $K_X + B^h$ and an effective exceptional divisor, the MMP terminates over η_Z by [Bir12, Theorem

1.8]. Thus we get a model (V, Δ_V) on which $K_V + \Delta_V \sim_{\mathbb{Q}} 0$ over η_Z , and hence over the generic point $\eta_{Z'}$ of Z' . Applying [Bir12, Theorem 1.4] or [HX13, Theorem 1.1], we can run an MMP on $K_W + \Delta_W$ over Z' ending with a good minimal model. Replacing V with the good minimal model of (W, Δ_W) over Z' , we can assume that $K_V + \Delta_V$ is semi-ample over Z' and induces a contraction $f' : V \rightarrow Z''/Z'$.

Step 2. Take a common resolution $\pi : Y \rightarrow X$ and $\pi' : Y \rightarrow V$. Then

$$P_Y := \pi'^*(K_V + \Delta_V) - \pi^*(K_X + B)$$

is vertical over Z'' . In addition, since $K_X + B \sim_{\mathbb{Q}} 0/Z$ and $K_V + \Delta_V \sim_{\mathbb{Q}} 0/Z''$, we conclude that $P_Y \sim_{\mathbb{Q}} 0/Z''$, which implies that P_Y is the pullback of an \mathbb{Q} -Cartier \mathbb{Q} -divisor $P_{Z''}$ by [CHL23, Lemma 2.5]. The adjunction formula

$$K_X + B \sim_q f^*(K_Z + B_Z + M_Z)$$

induces the following adjunction formula

$$(*) \quad K_V + \Delta_V \sim_q f'^*(K_{Z''} + B_{Z''} + P_{Z''} + M_{Z''}),$$

where $K_{Z''} + B_{Z''} + M_{Z''}$ is the pullback of $K_Z + B_Z + M_Z$ on Z'' . From (*) we see that the discriminant divisor $\Delta_{Z''}$ of (V, Δ_V) over Z'' is $B_{Z''} + P_{Z''}$ and the moduli divisor of $(X, B) \rightarrow Z$ on Z'' coincides with the moduli divisor of $(V, \Delta_V) \rightarrow Z''$.

Step 3. We claim that there exists $p \in \mathbb{N}$ depending only on d, q, u, Φ such that $pM_{Z''}$ is integral. Then $pM_{Z''}$ is integral and hence Cartier because Z'' is smooth.

As in the fifth paragraph of the proof of [Bir21a, Lemma 7.4], we can reduce to the case where Z and Z'' are curves. Then the claim follows from [Bir21a, Lemma 7.3]. Note that in [Bir21a, Lemma 7.3], A is an effective integral divisor on X . In our case, the coefficients of A are in a DCC set of positive real numbers. The proof of [Bir21a, Lemma 7.3] can still be applied with some slight changes. In Step 2 of the proof of [Bir21a, Lemma 7.3], replace [HX13, Theorem 1.1] with [Has19, Theorem 1.2], which is an \mathbb{R} -version of [HX13, Theorem 1.1], and in Step 5, replace [Bir23a, Theorem 1.7] with [Bir23a, Theorem 6.4].

□

3.2. Finite coefficients case. We first give an effective adjunction formula in the case that the coefficients of B are in a fixed finite set of real numbers.

Theorem 3.2. *Let $d \in \mathbb{N}$, $u \in \mathbb{R}^{>0}$, $I \subset \mathbb{R}^{\geq 0}$ be a finite set, and $\Phi \subset \mathbb{R}^{\geq 0}$ be a DCC set. Then there exists a finite set $J \subset \mathbb{R}^{\geq 0}$ depending only on d, u, I, Φ satisfying the following. Assume that*

- (X, B) is a projective lc pair of dimension d and $B \in I$,
- $f : X \rightarrow Z$ is a contraction with $K_X + B \sim_{\mathbb{R}} 0/Z$,
- (X, B) is klt over the generic point η_Z of Z ,
- $A \in \Phi$ is an effective \mathbb{R} -divisor on X such that A is relatively semi-ample over the generic point η_Z of Z , and
- $0 < \text{vol}(A|_F) \leq u$ for the general fiber F of f .

Then there exists an adjunction formula

$$K_X + B \sim_{\mathbb{R}} f^*(K_Z + B_Z + M_Z)$$

such that $M_{Z'} = \sum_{i=1}^l r_i M_{i,Z'}$ on some high resolution $Z' \rightarrow Z$, where $r_i \in J$ and $M_{i,Z'}$ is Cartier nef for any i .

Moreover, if in addition (X, B) is ϵ -lc for some $\epsilon \in \mathbb{R}^{>0}$, then the coefficients of B are in a fixed finite set Ψ depending only on d, u, ϵ, I, Φ .

Proof. Step 1. By [HLX23, Theorem 3.3], we can write

$$K_X + B = \sum_{i=1}^l r_i (K_X + B_i)$$

such that

- r_1, \dots, r_l belong to a finite set $J \subset \mathbb{R}^{\geq 0}$ depending only on d, I ,
- r_1, \dots, r_l are \mathbb{Q} -linear independent and $\sum_{i=1}^l r_i = 1$,
- (X, B_i) is lc and $\text{Nklt}(X, B_i) = \text{Nklt}(X, B)$ for any i (in particular, (X, B_i) is klt over the generic point η_Z of Z for any i),
- $K_X + B_i \sim_{\mathbb{Q}} 0/Z$ for any i ,
- there exists $q \in \mathbb{N}$ depending only on d, I such that $q(K_X + B_i)$ is integral for any i , and
- if \mathbf{M} and \mathbf{M}_i are the moduli part of the adjunction formulas with respect to (X, B) and (X, B_i) , then $\mathbf{M} = \sum_{i=1}^l r_i \mathbf{M}_i$.

Take F as a general fiber of f , then $K_F + B_{i,F} = (K_X + B_i)|_F \sim_{\mathbb{Q}} 0$. Hence $(F, B_{i,F}, A_F)$ is a polarized klt Calabi-Yau pair. Since the coefficients of $B_{i,F}$ are in a finite set and the coefficients of A_F are in a DCC set Φ , they are bounded from below away from zero. By [Bir19, Lemma 2.48], $(F, B_{i,F})$ is δ -lc for some positive real number δ depending only on d, I . Therefore, $(F, \text{Supp}(B_{i,F} + A_F))$ belongs to a bounded family depending only on δ, Φ, u by [Bir23a, Theorem 6.2]. Hence by [Bir21a, Lemma 7.2], possibly replacing q with a bounded multiple, we can assume that $q(K_F + B_{i,F}) \sim 0$. This implies that we can find a rational function α_i on X such that $q(K_X + B_i) + \text{Div}(\alpha_i)$ is vertical over Z_i . Since

$$q(K_X + B_i) + \text{Div}(\alpha_i) \sim_{\mathbb{Q}} 0/Z,$$

we see that $q(K_X + B_i) + \text{Div}(\alpha_i)$ is the pullback of a \mathbb{Q} -Cartier \mathbb{Q} -divisor qL_Z on Z by [CHL23, Lemma 2.5]. Thus we have the following adjunction formula

$$K_X + B_i \sim_q f^*(K_Z + B_{i,Z} + M_{i,Z})$$

where $B_{i,Z}$ is the discriminant divisor and $M_{i,Z} = L_Z - K_Z - B_{i,Z}$ is the moduli divisor.

By Lemma 2.10 and Lemma 3.1, there exists a positive integer p depending only on d, u, I, Φ such that $pM_{i,Z'}$ is Cartier nef on some high resolution $Z' \rightarrow Z$.

Step 2. Now we have an adjunction formula

$$(*) \quad K_X + B \sim_{\mathbb{R}} f^*(K_Z + B_Z + M_Z),$$

where $B_Z = \sum_{i=1}^l r_i B_{i,Z}$ and $\mathbf{M} = \sum_{i=1}^l r_i \mathbf{M}_i$. Let $J' = \{\frac{r}{p} | r \in J\}$, then

$$M_{Z'} = \sum_{i=1}^l \frac{r_i}{p} (pM_{i,Z'})$$

where $\frac{r_i}{p} \in J'$ and $pM_{i,Z'}$ is Cartier nef for $1 \leq i \leq l$.

Step 3. In this step we prove the moreover part. Now we assume that (X, B) is ϵ -lc for some fixed $\epsilon \in \mathbb{R}^{>0}$, then in Step 1, we can choose (X, B_i) to be $\frac{\epsilon}{2}$ -lc by [HLS24, Corollary 5.5]. By the argument of the proof of [BH22, Lemma 6.7] and Lemma 2.11, replacing q with a bounded multiple, we can assume that $qB_{i,Z}$ is integral. Since $B_Z = \sum_{i=1}^l r_i B_{i,Z}$ and r_i 's are in a finite set, we conclude that $B_Z \in \Psi$ where $\Psi \subset \mathbb{R}^{\geq 0}$ is a finite set. \square

3.3. DCC coefficients case. Next we show that there exists an effective adjunction formula in the case that the coefficients of B are in a fixed DCC set of real numbers.

Theorem 3.3. *Let $d \in \mathbb{N}$, $u \in \mathbb{R}^{>0}$ and $\Phi \subset \mathbb{R}^{\geq 0}$ be a DCC set. Then there exists a finite set $I \subset \mathbb{R}^{\geq 0}$ depending only on d, u, Φ satisfying the following. Assume that*

- (X, B) is a projective klt pair of dimension d and $B \in \Phi$,
- $f : X \rightarrow Z$ is a contraction with $K_X + B \sim_{\mathbb{R}} 0/Z$,
- $A \in \Phi$ is an effective \mathbb{R} -divisor on X such that A is relatively semi-ample over the generic point η_Z of Z , and
- $0 < \text{vol}(A|_F) \leq u$ for the general fiber F of f .

Then there is an adjunction formula

$$K_X + B \sim_{\mathbb{R}} f^*(K_Z + B_Z + M_Z)$$

such that $M_{Z'} = \sum_{i=1}^l r_i M_{i,Z'}$ on some high resolution $Z' \rightarrow Z$, where $r_i \in I$ and $M_{i,Z'}$ is Cartier nef for any i .

Proof. *Step 1.* Let F be a general fiber of $f : X \rightarrow Z$. Then $K_F + B_F := (K_X + B)|_F \sim_{\mathbb{R}} 0$ and (F, B_F) is a klt log Calabi-Yau pair. By global ACC for lc thresholds [HMX14, Theorem 1.5], since the coefficients of B_F are in a DCC set Φ , they are in a finite set $J \subset \mathbb{R}^{\geq 0}$ depending only on d, Φ . Hence if we denote B^h to be the horizontal/ Z part of B , then $B^h \in J$.

Step 2. In this step and the next step, we obtain a new pair from which preserves the horizontal part of B over the generic point of Z but changes the vertical part of B to be reduced.

Take high log resolutions of (X, B) and Z as follows:

$$\begin{array}{ccc} X' & \xrightarrow{\pi} & X \\ f' \downarrow & & \downarrow f \\ Z' & \xrightarrow{\mu} & Z \end{array}$$

such that (X', Σ) is log smooth, where Σ is the sum of reduced π -exceptional divisors and the birational transform of $\text{Supp } B$. Write $K_{X'} + B' = \pi^*(K_X + B)$. Let \tilde{B}^v, \tilde{B}^h be the vertical/ Z' part and horizontal/ Z' part of the birational transform of B . Let E^v, E^h be the vertical/ Z' part and horizontal/ Z' part of the reduced π -exceptional divisors. Then we take an open subset U' in Z' such that

- $\mu : Z' \rightarrow Z$ is an isomorphism on U' ,
- $L := Z' \setminus U'$ is a reduced divisor on Z' , and
- $f'(\text{Supp}(\tilde{B}^v + E^v)) \subseteq L$.

Let $f'^{-1}L$ be the reduction of the inverse image of L with respect to $f' : X' \rightarrow Z'$ and add $f'^{-1}L$ to Σ . Possibly replacing (X', Σ) with a higher birational model, we can assume that the condition (X', Σ) being log smooth is preserved.

Step 3. Let $\Gamma' = \tilde{B}^h + E^h + f'^{-1}L$. Replacing J with $J \cup \{1\}$, we have $\Gamma' \in J$. Run an MMP on $K_{X'} + \Gamma'$ over Z' with scaling of some ample divisor. Since over $f'^{-1}U'$, (X, B) is a weak lc model of (X', Γ') , hence by [Bir12, Corollary 3.7], (X', Γ') has a minimal model over U' . Therefore, by [Bir12, Theorem 1.9], the MMP terminates over $f'^{-1}U'$ and we reach a model (W, Γ_W) such that $K_W + \Gamma_W \sim_{\mathbb{R}} 0/U'$.

Now we continue to run the MMP on $K_W + \Gamma_W$ over Z' . The MMP does not modify W over U' . Moreover, the MMP is also an MMP on $K_W + \Gamma_W - aF_W$ where F_W is the pullback of L with respect to $W \rightarrow Z'$ and $a > 0$ is a small number. Since $K_W + \Gamma_W - aF_W$ is semi-ample over U' and $(W, \Gamma_W - aF_W)$ is klt, the MMP terminates with a good minimal model V by [Has19, Theorem 1.2]. Let $g : V \rightarrow Z''$ be the contraction induced by the semi-ample/ Z' \mathbb{R} -divisor $K_V + \Gamma_V$ and denote by μ' the morphism $Z'' \rightarrow Z'$. If we denote $K_V + B_V$ as the pushdown of $K_{X'} + B'$, then $\text{Supp}(\Gamma_V - B_V)$ maps into $L \subseteq Z'$. Since $K_{X'} + B' \sim_{\mathbb{R}} 0/Z'$, by the cone theorem, the pullbacks of $K_V + B_V$ and $K_X + B$ to a common resolution are the same. Therefore, we conclude that (V, B_V) is a sub-klt pair and $K_V + B_V \sim_{\mathbb{R}} 0/Z$. Let A_V be the birational transform of the horizontal/ Z part of A . Let G be the general fiber of $g : V \rightarrow Z''$. Since over U' , (V, Γ_V) is a small \mathbb{Q} -factorialization of (X, B) , A_V is the pullback of A . Therefore, A_V is relatively semi-ample over the generic point of Z'' and $0 < \text{vol}(A_V|_G) \leq u$.

Step 4. Applying Theorem 3.2 to (V, Γ_V) over Z'' , there exists a finite set $I \subset \mathbb{R}^{\geq 0}$ depending only on d, u, J, Φ such that we can write an adjunction formula

$$(*) \quad K_V + \Gamma_V \sim_{\mathbb{R}} g^*(K_{Z''} + \Gamma_{Z''} + M_{Z''})$$

such that $M_{Z''} = \sum_{i=1}^l r_i M_{i, Z''}$ on some high resolution $Z''' \rightarrow Z''$, where $r_i \in I$ and $M_{i, Z''}$ is Cartier nef for any i .

Since $K_V + \Gamma_V$ is the pullback of $K_X + B$ over $U' \subseteq Z'$, $\Gamma_V - B_V$ is vertical over Z'' . Since

$$K_V + \Gamma_V \sim_{\mathbb{R}} K_V + B_V \sim_{\mathbb{R}} 0/Z'',$$

we conclude that $\Gamma_V - B_V$ is the pullback of an effective \mathbb{R} -Cartier \mathbb{R} -divisor $P_{Z''}$ on Z'' by [Li24b, Lemma 2.11]. The adjunction formula $(*)$ induces an adjunction

formula

$$(**) \quad K_V + B_V \sim_{\mathbb{R}} g^*(K_{Z''} + B_{Z''} + M_{Z''})$$

where $B_{Z''} := \Gamma_{Z''} - P_{Z''}$.

Since the pullbacks of $K_V + B_V$ and $K_X + B$ to a common resolution are the same, the adjunction formula (**) induces the following adjunction formula

$$(***) \quad K_X + B \sim_{\mathbb{R}} f^*(K_Z + B_Z + M_Z)$$

where $K_Z + B_Z + M_Z$ is the pushdown of $K_{Z''} + B_{Z''} + M_{Z''}$. □

4. SINGULARITIES OF KLT STABLE MINIMAL MODELS

In this section we apply the effective adjunction formula to prove Theorem 1.6. We follow the proof of [Bir21a, Lemma 8.2].

Proof of Theorem 1.6. *Step 1.* Let $(X, B), A \in \mathcal{S}_{klt}(d, \Phi, \leq u, v)$ and $f : X \rightarrow Z$ be the contraction defined by the semi-ample \mathbb{R} -divisor $K_X + B$. By Theorem 3.3, there exists a finite set $I \subset \mathbb{R}^{\geq 0}$ depending only on d, u, Φ such that there is an adjunction formula

$$(*) \quad K_X + B \sim_{\mathbb{R}} f^*(K_Z + B_Z + M_Z)$$

satisfying that $M_{Z'} = \sum_{i=1}^l r_i M_{i,Z'}$ on some high resolution $Z' \rightarrow Z$, where $r_i \in I$ and $M_{i,Z'}$ is Cartier nef for any i . By the ACC for lc thresholds [HMX14, Theorem 1.1], the coefficients of B_Z are in a DCC set $\Psi \subset \mathbb{R}^{\geq 0}$ depending only on d, Φ . Since

$$\text{vol}(K_Z + B_Z + M_Z) = \text{Ivol}(K_X + B) = v,$$

we conclude that $(Z, B_Z + M_Z) \in \mathcal{F}_{gklt}(\dim Z, \Psi \cup I, v)$. By [Bir21a, Theorem 1.5], $(Z, B_Z + M_Z)$ is generalized δ -lc for some $\delta \in \mathbb{R}^{>0}$ depending only on $\dim Z, \Psi, I, v$, hence depending only on d, Φ, u, v .

Step 2. If D is a prime divisor over X which is horizontal over Z , then D determines a prime divisor S over the general fiber F of f . Since (F, B_F) is a klt log Calabi-Yau pair and $B_F \in \Phi$, by [Bir19, Lemma 2.48], (F, B_F) is τ -lc for some $\tau \in \mathbb{R}^{>0}$ depending only on $\dim F$ and Φ , hence depending only on d, Φ . Then

$$a(D, X, B) = a(S, F, B_F) \geq \tau.$$

Step 3. If D is a prime divisor over X which is vertical over Z , then we take high resolutions as follows:

$$\begin{array}{ccc} X' & \xrightarrow{\pi} & X \\ f' \downarrow & & \downarrow f \\ Z' & \xrightarrow{\mu} & Z \end{array}$$

such that D is a divisor on X' and its image on Z' is a prime divisor E . Let

$$K_{X'} + B' = \pi^*(K_X + B)$$

and

$$K_{Z'} + B_{Z'} + M_{Z'} = \mu^*(K_Z + B_Z + M_Z).$$

Since $(Z, B_Z + M_Z)$ is generalized δ -lc,

$$a(E, Z', B_{Z'} + M_{Z'}) = a(E, Z, B_Z + M_Z) \geq \delta.$$

Therefore,

$$\text{mult}_E B_{Z'} \leq 1 - \delta.$$

By the definition of discriminant divisors, $(X', B' + \delta f'^*E)$ is sub-lc over the generic point of E . This implies that

$$\text{mult}_D B' + \delta \text{mult}_D f'^*E \leq 1$$

and hence $\text{mult}_D B' \leq 1 - \delta$. Thus

$$a(D, X, B) = a(D, X', B') \geq \delta.$$

Step 4. From the above arguments we see that (X, B) is ϵ -lc, where $\epsilon := \min\{\tau, \delta\}$ depending only on d, Φ, u, v . □

5. BOUNDEDNESS OF KLT STABLE MINIMAL MODELS

5.1. Lower bound on lc thresholds of strongly stable minimal models. We prove there is a uniform lower bound on lc thresholds of strongly stable minimal models with klt singularities, which is the \mathbb{R} -coefficient version of [Bir21a, Lemma 8.3] and the proof is similar to the one there. The new tools applied here are the effective adjunction formula with \mathbb{R} -coefficients (Theorem 3.3) and uniform decomposition of generalized \mathbb{R} -pairs (Lemma 2.16).

Theorem 5.1. *Let $d \in \mathbb{N}$, $u, v, w \in \mathbb{R}^{>0}$, and $\Phi \subset \mathbb{R}^{\geq 0}$ be a DCC set. Then there exists $\lambda \in \mathbb{R}^{>0}$ depending only on d, u, v, w, Φ such that for any strongly stable minimal model*

$$(X, B), A \in \mathcal{SS}_{\text{klt}}(d, \Phi, \leq u, v)$$

with $\text{vol}(K_X + B + A) \leq w$, the pair $(X, B + \lambda A)$ is lc.

Proof. *Step 1.* Let $(X, B), A \in \mathcal{SS}_{\text{klt}}(d, \Phi, \leq u, v)$. By Theorem 1.6, there is $\epsilon \in \mathbb{R}^{>0}$ depending only on d, u, v, Φ such that (X, B) is ϵ -lc. Since $K_X + B \sim_{\mathbb{R}} 0/Z$, we have $K_F + B_F = (K_X + B)|_F \sim_{\mathbb{R}} 0$, where F is the general fiber of f . Since A is ample over Z , $(F, B_F), A_F$ is a polarized ϵ -lc log Calabi-Yau pair. Since the coefficients of B_F, A_F are in a DCC set Φ , they are bounded from below away from zero.

Step 2. Let $s = \dim Z$. If $s = 0$, then $(X, B), A$ is bounded by [Bir23a, Theorem 6.2], in this case the theorem follows from [Bir21b, Theorem 1.8]. Now we assume that $s \geq 1$. By Theorem 3.3, there exists a finite set $I \subset \mathbb{R}^{\geq 0}$ depending only on d, u, Φ such that there is an adjunction formula

$$K_X + B \sim_{\mathbb{R}} f^*(K_Z + B_Z + M_Z)$$

satisfying that $M_{Z'} = \sum_{i=1}^l \mu_i M_{i,Z'}$ on some high resolution $Z' \rightarrow Z$, where $\mu_i \in I$ and $M_{i,Z'}$ is Cartier nef for any i . By the ACC for lc thresholds [HMX14, Theorem 1.1], the coefficients of B_Z are in a DCC set $\Psi \subset \mathbb{R}^{\geq 0}$ depending only on d, Φ . Hence $(Z, B_Z + M_Z) \in \mathcal{F}_{gklt}(s, \Psi \cup I, v)$. By [Bir21a, Theorem 1.5], there is a fixed finite set $I' \subset \mathbb{R}^{> 0}$ such that $B_Z \in I'$. Therefore, $(Z, B_Z + M_Z) \in \mathcal{F}_{gklt}(s, I \cup I', v)$.

By Theorem 2.16, there is a fixed finite set $J \subset \mathbb{R}^{\geq 0}$ and a fixed $p \in \mathbb{N}$ such that we can decompose $K_Z + B_Z + M_Z$ as follows:

$$K_Z + B_Z + M_Z = \sum_{i=1}^l r_i (K_Z + B_{i,Z} + M_{i,Z})$$

satisfying that

- $\sum_{i=1}^l r_i = 1$ and $r_i \in J$ for any i ,
- $(Z, B_{i,Z} + M_{i,Z})$ is generalized klt for any i , and
- $p(K_Z + B_{i,Z} + M_{i,Z})$ is very ample for any i .

Step 3. Let

$$H := \sum_{i=1}^l p(K_Z + B_{i,Z} + M_{i,Z})$$

and let n be a fixed positive integer such that $n > \max\{\frac{p}{r_1}, \dots, \frac{p}{r_l}\}$, then $f^*H - (K_X + B)$ and $n(K_X + B) - f^*H$ are nef. Take $\pi : Y \rightarrow X$ as a \mathbb{Q} -factorialization of X and write $K_Y + B_Y = \pi^*(K_X + B)$. Since $(X, B), A$ is a strongly stable minimal model, $K_X + B + A$ is ample, thus $f^*H + A$ is ample, which implies that $\pi^*f^*H + \pi^*A$ is nef and big. Since Y is ϵ -lc and since

$$\pi^*f^*H + \pi^*A - K_Y = \pi^*(f^*H - K_X - B) + \pi^*A + B_Y$$

is pseudo-effective and the coefficients of π^*A are in the DCC set Φ , by [Bir23a, Theorem 4.2], there is a bounded positive integer m such that $|m(\pi^*f^*H + \pi^*A)|$ is birational, hence $|m(f^*H + A)|$ is birational.

Step 4. Replacing m with a bounded multiple we can assume that $mA \geq 1$. Pick a member

$$N \in |m(f^*H + A)|$$

and then

$$\begin{aligned} \text{vol}(N) &= \text{vol}(m(f^*H + A)) \\ &\leq \text{vol}(m(n(K_X + B) + A)) \\ &\leq (mn)^d w. \end{aligned}$$

For any component D of N , we have $\text{mult}_D(N + B + mA) \geq 1$ because any component of N whose coefficient is not an integer is a component of A . By construction,

$$N - (K_X + B + mA) \sim (m-1)f^*H + (f^*H - K_X - B)$$

is pseudo-effective. Applying [Bir19, Proposition 4.4] to $(X, B + mA), N$, we conclude that there is $c \in \mathbb{R}^{> 0}$ and a bounded set \mathcal{P} depending only on d, m, n, w such that

there is a log smooth couple $(\overline{X}, \overline{\Sigma}) \in \mathcal{P}$ and a birational map $\overline{X} \dashrightarrow X$ satisfying that

- $\overline{\Sigma}$ contains the reduced exceptional divisor of $\overline{X} \dashrightarrow X$ and the support of the birational transform of $B + A + N$, and
- if \overline{N} is the pullback of N to \overline{X} , then $\overline{N} \cdot \overline{H}^{d-1} \leq c$ for some very ample divisor $\overline{H} \leq \overline{\Sigma}$.

The meaning of the pullback of N to \overline{X} (and similarly for other divisors) is pulling back N to a common resolution of X, \overline{X} and then pushing down to \overline{X} . Note that in [Bir19, Proposition 4.4], N is required to be a \mathbb{Q} -divisor, but the proof of [Bir19, Proposition 4.4] goes through verbatim when N is an \mathbb{R} -divisor. Let \overline{A} be the pullback of A to \overline{X} , then $\overline{A} \cdot \overline{H}^{d-1} \leq c$ since $N - A$ is pseudo-effective, and this implies that $\overline{A} \leq c$.

Step 5. Let $K_{\overline{X}} + \overline{B}$ be the pullback of $K_X + B$ to \overline{X} , then $\overline{B} \leq 1 - \epsilon$ because (X, B) is ϵ -lc. Therefore, there is a fixed number $\lambda \in (0, 1)$ such that $(\overline{X}, \overline{B} + \lambda \overline{A})$ is sub-lc as $(\overline{X}, \overline{B} + \overline{A})$ is log smooth.

Since

$$K_X + B + \lambda A = \lambda(K_X + B + A) + (1 - \lambda)(K_X + B)$$

is ample, by negativity lemma we conclude that $(X, B + \lambda A)$ is lc. \square

5.2. Proof of Theorem 1.5. We are now ready to prove the boundedness of stable minimal models with klt singularities. As explained in the sketch of the proofs, given a stable minimal model $(X, B), A$ in Theorem 1.5, the key point is to shrink $\text{Nlc}(X, B + \lambda A)$ by induction.

Proof of Theorem 1.5. *Step 1.* Let $(X, B), A \in \mathcal{S}_{klt}(d, \Phi, \leq u, v)$ such that $(K_X + B)^i \cdot A^{d-i} \leq w$ for $0 \leq i \leq d$, and $f : X \rightarrow Z$ be the contraction induced by the semi-ample \mathbb{R} -divisor $K_X + B$. By Theorem 1.6, there exists $\epsilon \in \mathbb{R}^{>0}$ depending only on d, Φ, u, v such that (X, B) is ϵ -lc.

Since $K_X + B \sim_{\mathbb{R}} 0/Z$, we have $K_F + B_F = (K_X + B)|_F \sim_{\mathbb{R}} 0$, where F is the general fiber of f . Thus $(F, B_F), A_F$ is a polarized ϵ -lc log Calabi-Yau pair. Since the coefficients of A, B are in a DCC set Φ , they are bounded from below away from zero. Therefore, by [Bir23a, Theorem 6.2], $(F, \text{Supp}(B_F + A_F))$ belongs to a bounded family.

Step 2. Let $s = \dim Z$. If $s = 0$, then by Step 1, $(X, B), A$ is bounded, hence we can assume that $s \geq 1$. By Theorem 3.3, there exists a finite set $I \subset \mathbb{R}^{\geq 0}$ depending only on d, u, Φ such that there is an adjunction formula

$$K_X + B \sim_{\mathbb{R}} f^*(K_Z + B_Z + M_Z)$$

satisfying that $M_{Z'} = \sum_{i=1}^l \mu_i M_{i, Z'}$ on some high resolution $Z' \rightarrow Z$, where $\mu_i \in I$ and $M_{i, Z'}$ is Cartier nef for any i . By the ACC for lc thresholds [HMX14, Theorem 1.1], the coefficients of B_Z are in a DCC set $\Psi \subset \mathbb{R}^{\geq 0}$. Hence $(Z, B_Z + M_Z) \in \mathcal{F}_{gklt}(s, \Psi \cup I, v)$. By [Bir21a, Theorem 1.5], there is a finite set $I' \subset \mathbb{R}^{\geq 0}$ depending

only on s, v, Ψ, I , hence depending only on d, u, v, Φ , such that $B_Z \in I'$. Therefore, $(Z, B_Z + M_Z) \in \mathcal{F}_{gklt}(s, I \cup I', v)$.

By Theorem 2.16, there is a fixed finite set $J \subset \mathbb{R}^{\geq 0}$ and a fixed $p \in \mathbb{N}$ such that we can decompose $K_Z + B_Z + M_Z$ as follows:

$$K_Z + B_Z + M_Z = \sum_{i=1}^l r_i (K_X + B_{i,Z} + M_{i,Z})$$

satisfying that

- $\sum_{i=1}^l r_i = 1$ and $r_i \in J$ for any i ,
- $(Z, B_{i,Z} + M_{i,Z})$ is generalized klt for any i , and
- $L_i := p(K_Z + B_{i,Z} + M_{i,Z})$ is a very ample for any i .

Step 3. Define $P(t)$ as the image of non-lc locus of $(X, B + tA)$ on Z , i.e.

$$P(t) := f(\text{Nlc}(X, B + tA)).$$

We regard $P(t)$ as a closed subset of Z . Now we prove the following statement \mathcal{C}_k by induction on k : there exists $\lambda_k \in \mathbb{R}^{>0}$ depending only on d, Φ, u, v, w such that

- $\dim P(\lambda_k) \leq s - k$, and
- $K_X + B + \lambda_k A$ is nef in dimension $k - 1$ over Z .

By [Bir21b, Theorem 1.8], there exists $\lambda_1 \in \mathbb{R}^{>0}$ such that $(F, B_F + \lambda_1 A_F)$ is lc. Then $(X, B + \lambda_1 A)$ is lc over some open subset $U \subseteq Z$, hence $\dim P(\lambda_1) \leq s - 1$. Since A is relatively nef over Z and $K_X + B$ is semi-ample, we conclude that $K_X + B + \lambda_1 A$ is nef in dimension 0 over Z .

Assume that the statement \mathcal{C}_k holds and λ_k is chosen for \mathcal{C}_k . We prove the statement \mathcal{C}_{k+1} in the following steps.

Step 4. In this step we prove that there exists a fixed $\lambda_{k+1} \in \mathbb{R}^{>0}$ such that $K_X + B + \lambda_{k+1} A$ is nef in dimension k over Z .

Let H_1, \dots, H_{s-k} be any very ample divisors in Z . Take a general member of the linear system $|H_i|$ and in abuse of notation, we still denote by H_i the general member. Let $T = \cap_{i=1}^{s-k} H_i$, $R_i = f^{-1} H_i$ and $S = \cap_{i=1}^{s-k} R_i = f^{-1} T$. We can write

$$K_S + B_S + \lambda_k A_S = (K_X + B + \lambda_k A + \sum_{i=1}^{s-k} R_i)|_S,$$

where $A_S = A|_S$.

If $K_S + B_S + \lambda_k A_S$ is nef, then take $\lambda_{k+1} = \lambda_k$ and we are done. Now assume that $K_S + B_S + \lambda_k A_S$ is not nef. By [FH23, Theorem 1.2, Corollary 1.4],

$$\text{Nlc}(S, B_S + \lambda_k A_S) = S \cap \text{Nlc}(X, B + \lambda_k A).$$

Since $\dim P(\lambda_k) \leq s - k$, $S = \cap_{i=1}^{s-k} f^{-1} H_i$, and H_i are general hypersurfaces in Z , we deduce that $\text{Nlc}(S, B_S + \lambda_k A_S)$ is contained in finitely many fibers of $S \rightarrow T$. Let R be a $(K_S + B_S + \lambda_k A_S)$ -negative extremal ray. Since $K_S + B_S + \lambda_k A_S$ is ample over T and $K_S + B_S$ is the pullback of an ample divisor on T , $K_S + B_S$ is positive on

$R \setminus \{0\}$, which implies that R is not contained in the image of the map of the closed cone of curves

$$\overline{NE}(\mathrm{Nlc}(S, B_S + \lambda_k A_S)) \rightarrow \overline{NE}(S).$$

By the length of extremal ray [Fuj11, Theorem 1.1(5)], there is a curve C on S generating R and satisfying that

$$(K_S + B_S + \lambda_k A_S) \cdot C \geq -2(d - s + k).$$

Take a fixed positive integer

$$m > \max\left\{\frac{p}{r_1}, \dots, \frac{p}{r_l}\right\},$$

then $m(K_Z + B_Z + M_Z) - L_i$ is ample and thus $m(K_X + B) - f^*L_i$ is nef. Therefore,

$$m(K_X + B)|_S \cdot C \geq f^*L_1|_S \cdot C = L_1 \cdot f(C) \geq 1$$

because L_1 is a very ample Cartier divisor on Z . Now we have

$$\begin{aligned} & \left((1 + 2(d - s + k)m)(K_X + B + \sum_{i=1}^{s-k} R_i)|_S + \lambda_k A_S \right) \cdot C \\ & \geq (K_S + B_S + \lambda_k A_S) \cdot C + 2(d - s + k)m(K_X + B)|_S \cdot C \geq 0. \end{aligned}$$

Let

$$\lambda_{k+1} = \frac{\lambda_k}{1 + 2(d - s + k)m}.$$

The above argument implies that $K_S + B_S + \lambda_{k+1} A_S$ is nef. Therefore, $K_X + B + \lambda_{k+1} A$ is nef in dimension k over Z .

Step 5. Let H_1, \dots, H_{s-k} be general members of one of the linear systems $|L_i|$ (each H_j can be in different linear systems). Recall that $m(K_X + B) - f^*L_i$ is nef for $1 \leq i \leq l$, thus $m(s - k)(K_X + B) - \sum_{i=1}^{s-k} f^*H_i$ is nef. By Lemma 2.18, we have the following inequality:

$$(*) \quad ((1 + m(s - k))(K_X + B) + \lambda_{k+1} A)^{d-s+k} \cdot f^*H_1 \cdots f^*H_{s-k} \geq 0.$$

Step 6. In this step we prove that after replacing λ_{k+1} with a smaller number, $\dim P(\lambda_{k+1}) \leq s - k - 1$ and hence \mathcal{C}_{k+1} is satisfied.

Let Q_1, \dots, Q_{s-k} be general members of the linear system $|L_1|$. Abusing notation denote $T = \cap_{i=1}^{s-k} Q_i$, $R_i = f^*Q_i$ and $S = \cap_{i=1}^{s-k} R_i = f^{-1}T$. We can write

$$K_S + B_S + \lambda_{k+1} A_S = (K_X + B + \lambda_{k+1} A + \sum_{i=1}^{s-k} R_i)|_S,$$

where $A_S = A|_S$.

- Since $K_X + B + \lambda_{k+1} A$ is nef in dimension k over Z , $K_S + B_S + \lambda_{k+1} A_S$ is nef. We claim that $K_S + B_S + tA_S$ is ample for any $t \in (0, \lambda_{k+1})$: Fix $t \in (0, \lambda_{k+1})$. Since $K_S + B_S$ is the pullback of an ample divisor on T and A_S is ample over T , there is a sufficiently small $0 < t' < t$ such that $K_S + B_S + t'A_S$ is ample. Then $K_S + B_S + tA_S$ is a positive linear combination of $K_S + B_S + t'A_S$ and $K_S + B_S + \lambda_{k+1} A_S$, which implies that $K_S + B_S + tA_S$ is ample.

Therefore, replacing λ_{k+1} with $\frac{\lambda_{k+1}}{2}$ we can assume that $K_S + B_S + \lambda_{k+1}A_S$ is ample. Note that the condition $K_X + B + \lambda_{k+1}A$ being nef in dimension k over Z is preserved.

- Let \bar{F} be the general fiber of $S \rightarrow T$ and F be the general fiber of $X \rightarrow Z$. Since T is the complete intersection of general hypersurfaces,

$$\text{vol}(A_S|_{\bar{F}}) = \text{vol}(A|_F) \leq u.$$

- Note that

$$\left(\sum_{i=1}^l r_i L_i\right)^s = \text{vol}(p(K_Z + B_Z + M_Z)) = p^s v.$$

If $(\alpha_1, \dots, \alpha_l)$ is an element of the set

$$\{(n_1, \dots, n_l) | n_i \in \mathbb{N}, \sum_{i=1}^l n_i = s\},$$

then the intersection number

$$L_1^{\alpha_1} \cdots L_l^{\alpha_l}$$

is a positive integer and bounded from above. Without loss of generality, we can assume that

$$L_1^{\alpha_1} \cdots L_l^{\alpha_l}$$

are fixed numbers for all

$$(\alpha_1, \dots, \alpha_l) \in \{(n_1, \dots, n_l) | n_i \in \mathbb{N}, \sum_{i=1}^l n_i = s\}.$$

Therefore,

$$\begin{aligned} & \text{Ivol}(K_S + B_S) \\ &= \text{Ivol}\left((K_X + B + \sum_{i=1}^{s-k} R_i)|_S\right) \\ &= \text{vol}\left((K_Z + B_Z + M_Z + \sum_{i=1}^{s-k} Q_i)|_T\right) \\ &= \left(\frac{1}{p} \sum_{i=1}^l r_i L_i + (s-k)L_1\right)^k \cdot L_1^{s-k} \end{aligned}$$

is a fixed positive real number, say v_k .

- Now we show that $\text{vol}(K_S + B_S + \lambda_{k+1}A_S)$ is bounded from above.

$$\begin{aligned} & \text{vol}(K_S + B_S + \lambda_{k+1}A_S) \\ &= \text{vol}\left((K_X + B + \lambda_{k+1}A + \sum_{i=1}^{s-k} R_i)|_S\right) \end{aligned}$$

$$\begin{aligned}
&\leq \text{vol}(((1 + m(s - k))(K_X + B) + \lambda_{k+1}A)|_S) \\
&= ((1 + m(s - k))(K_X + B) + \lambda_{k+1}A)^{d-s+k} \cdot (f^*L_1)^{s-k} \\
&\leq ((1 + m(s - k))(K_X + B) + \lambda_{k+1}A)^{d-s+k} \cdot (m(K_X + B))^{s-k}
\end{aligned}$$

is bounded from above because $(K_X + B)^i \cdot A^{d-i} \leq w$ for $0 \leq i \leq d$. The first inequality follows from the nefness of $m(s - k)(K_X + B) - \sum_{i=1}^{s-k} R_i$. The second inequality follows from the inequality (*) in Step 5.

The above argument implies that

$$(S, B_S), \lambda_{k+1}A_S \in \mathcal{SS}_{klt}(d - s + k, \Phi \cup \lambda_{k+1}\Phi, \leq \lambda_{k+1}^{d-s}u, v_k).$$

Hence we can apply Theorem 5.1 to $(S, B_S), \lambda_{k+1}A_S$. Replacing λ_{k+1} with a smaller fixed number, we can assume that $(S, B_S + \lambda_{k+1}A_S)$ is lc. By inversion of adjunction [Hac14, Theorem 0.1] or [Kaw07], we conclude that $(X, B + \lambda_{k+1}A)$ is lc near S . Since $S = f^{-1}T$ and T is the complete intersection of $(s - k)$ general hypersurfaces, we deduce that $\dim P(\lambda_{k+1}) \leq s - k - 1$, hence \mathcal{C}_{k+1} is satisfied.

Step 7. In this step we prove that there exists a fixed $\lambda \in \mathbb{R}^{>0}$ such that $(X, B + \lambda A)$ is lc and $K_X + B + \lambda A$ is globally nef.

By induction on k , \mathcal{C}_s is satisfied for some fixed $\lambda_s \in \mathbb{R}^{>0}$. Then $(X, B + \lambda_s A)$ is lc over the complement of a finite set of closed points of Z because $\dim P(\lambda_s) \leq 0$.

If $K_X + B + \lambda_s A$ is nef, then let $\lambda = \lambda_s$. If $K_X + B + \lambda_s A$ is not nef, let R be a $(K_X + B + \lambda_s A)$ -negative extremal ray. Repeating the process in Step 4, there is a curve C generating R and satisfying that

$$(K_X + B + \lambda_s A) \cdot C \geq -2d$$

and

$$m(K_X + B) \cdot C \geq 1.$$

Hence

$$((1 + 2md)(K_X + B) + \lambda_s A) \cdot R \geq 0.$$

This argument implies that

$$(1 + 2md)(K_X + B) + \lambda_s A$$

is nef. Let $\lambda = \frac{\lambda_s}{1+2md}$, then $K_X + B + \lambda A$ is nef.

The same argument as in Step 6 implies that after replacing λ with $\frac{\lambda}{2}$, $K_X + B + \lambda A$ is ample. Therefore,

$$(X, B), \lambda A \in \mathcal{SS}_{klt}(d, \Phi \cup \lambda\Phi, \leq \lambda^{d-s}u, v).$$

Moreover,

$$\text{vol}(K_X + B + \lambda A) = (K_X + B + \lambda A)^d$$

is bounded from above because $(K_X + B)^i \cdot A^{d-i} \leq w$ for $0 \leq i \leq d$. By Theorem 5.1, after replacing λ with a smaller fixed number, we can assume that $(X, B + \lambda A)$ is lc.

Step 8. In this step we finish the proof. Since (X, B) is ϵ -lc, $(X, B + \frac{\lambda}{2}A)$ is $\frac{\epsilon}{2}$ -lc. Since the coefficients of $B + \frac{\lambda}{2}A$ is in the DCC set $\Phi \cup \frac{\lambda}{2}\Phi$, there is a fixed $N \in \mathbb{N}$ such that $|N(K_X + B + \frac{\lambda}{2}A)|$ is birational by [HMX14, Theorem 1.3]. Thus $|K_X + (2d + 1)N(K_X + B + \frac{\lambda}{2}A)|$ is birational by [HMX13, Lemma 2.3.4]. Since $\text{vol}(K_X + B + \frac{\lambda}{2}A)$ is bounded from above, we conclude that $(X, B + \frac{\lambda}{2}A)$ is log birationally bounded by [HMX13, Theorem 3.1]. Note that $K_X + B + \frac{\lambda}{2}A$ is ample, thus $(X, B + \frac{\lambda}{2}A)$ and hence $(X, B), A$ belongs to a bounded family by [HMX14, Theorem 1.6]. \square

Remark 5.2. The condition “ $\text{vol}(A|_F) \leq u$ ” in Theorem 1.5 can be removed: apply [Bir22, Lemma 4.12], we have

$$(K_X + B)^{\dim Z} \cdot A^{d - \dim Z} = \text{Ivol}(K_X + B) \text{vol}(A|_F),$$

thus $\text{vol}(A|_F) \leq \frac{w}{v}$.

5.3. Proof of Theorem 1.7. In this subsection we first generalize [Fil20, Theorem 1.3] to the generalized pair case. Then we apply Theorem 1.5 to prove Theorem 1.7.

Lemma 5.3. *Let $d \in \mathbb{N}$, $\epsilon, v \in \mathbb{R}^{>0}$, and $I \subset \mathbb{R}^{\geq 0}$ be a finite set. Then there exists a finite set $J \subset \mathbb{R}^{\geq 0}$ depending only on d, ϵ, v, I satisfying the following.*

If $(X, B + M) \in \mathcal{F}_{gklt}(d, I, \leq v)$ is a generalized ϵ -lc pair, then $\text{vol}(K_X + B + M) \in J$.

Proof. We follow the proof of [Fil20, Theorem 1.3].

Step 1. Pick a generalized ϵ -lc pair $(X, B + M) \in \mathcal{F}_{gklt}(d, I, \leq v)$ with data $X' \rightarrow X$ and M' . Write $M' = \sum_{i=1}^l \mu_i M'_i$, where $\mu_i \in I$ and M'_i is Cartier nef for any i . By [Bir21a, Theorem 1.2] and its proof, there exists a bounded set of couples \mathcal{P} depending only on d, I, v such that there exists a log smooth couple $(\bar{X}, \bar{\Sigma}) \in \mathcal{P}$ and a birational map $\pi : \bar{X} \dashrightarrow X$ satisfying the following:

- $\bar{\Sigma}$ contains the exceptional divisors of π and the support of the birational transform of B ,
- each M'_i descends to \bar{X} as \bar{M}_i , and
- $\bar{A} - \sum_{i=1}^l \mu_i \bar{M}_i$ is pseudo-effective for some very ample divisor $\bar{A} \leq \bar{\Sigma}$.

In particular, M' descends to \bar{X} as $\bar{M} = \sum_{i=1}^l \mu_i \bar{M}_i$ and the number of M_i is bounded from above.

Let $\bar{B} = (1 - \epsilon)E + \tilde{B}$, where E is the sum of exceptional divisors of π and \tilde{B} is the strict transform of B . Since $K_X + B + M$ is ample, by negativity lemma, if D is a prime divisor on X which is exceptional over \bar{X} , then

$$1 \geq a(D, X, B + M) \geq a(D, \bar{X}, \bar{B} + \bar{M}) \geq a(D, \bar{X}, \bar{B}).$$

Since (\bar{X}, \bar{B}) is ϵ -lc and belongs to a bounded family, we can extract all such D and obtain a birational model $(\bar{X}', \bar{\Sigma}')$ which is also in a bounded family by [Bir24, Theorem 1.2], where $\bar{\Sigma}'$ is the sum of all exceptional divisors of $\bar{X}' \rightarrow \bar{X}$ and the strict transform of $\bar{\Sigma}$. Replacing $(\bar{X}, \bar{\Sigma})$ with a log bounded resolution of $(\bar{X}', \bar{\Sigma}')$, we can assume that $\pi^{-1} : X \dashrightarrow \bar{X}$ does not contract divisors. We still denote

$\bar{B} = (1 - \epsilon)E + \tilde{B}$, where E is the sum of exceptional divisors of π and \tilde{B} is the strict transform of B .

Step 2. Applying Noetherian induction, we can assume that there is a log smooth couple $(\bar{V}, \bar{\Gamma})$ and a smooth projective morphism $\bar{V} \rightarrow T$ onto a smooth variety, such that there is a closed point $t \in T$ so that we can identify \bar{X} with \bar{V}_t and $\bar{\Sigma} \leq \bar{\Gamma}_t$. Possibly taking a finite base change and shrinking T we can assume that $(\bar{V}, \bar{\Gamma})$ is log smooth over T .

Since the coefficients of B are in a finite set I , without loss of generality, we can assume that the number of components of B and their coefficients are fixed. Hence there exists a boundary $\bar{\Delta}$ on \bar{V} such that we can identify $\bar{\Delta}_t$ with \bar{B} .

By the argument of Step 3 and Step 4 in the proof of [Bir21a, Theorem 1.3], there exist divisors \bar{N}_i on \bar{V} such that $\bar{N}_i|_{\bar{V}_t} \sim_{\mathbb{Q}} \bar{M}_i$. If we write $\bar{N} = \sum_{i=1}^l \mu_i \bar{N}_i$, then $\bar{N}|_{\bar{V}_t} \sim_{\mathbb{R}} \bar{M}$. Since μ_i 's belong to a finite set I , without loss of generality, we can assume that μ_i 's are all fixed. Therefore, we regard \bar{N} as a fixed divisor on \bar{V} .

Step 3. Applying [Fil18, Theorem 1.12] to $(\bar{V}, \bar{\Delta}), \bar{N}$, we deduce that $\text{vol}(K_{\bar{X}} + \bar{B} + \bar{M})$ is fixed. Since $(X, B + M)$ is generalized ϵ -lc, $\bar{B} = (1 - \epsilon)E + \tilde{B}$ and $X \dashrightarrow \bar{X}$ does not contract divisors, we have

$$\text{vol}(K_{\bar{X}} + \bar{B} + \bar{M}) = \text{vol}(K_X + B + M).$$

Therefore, $\text{vol}(K_X + B + M)$ is fixed and we finish the proof. \square

Proof of Theorem 1.7. Let $(X, B), A \in \mathcal{S}_{klt}(d, \Phi, \leq u, \leq v)$ such that (X, B) is ϵ -lc, and let $f : X \rightarrow Z$ be the contraction induced by the semi-ample \mathbb{R} -divisor $K_X + B$. By Theorem 3.2, there exist finite sets $J, \Psi \subset \mathbb{R}^{\geq 0}$ depending only on d, u, ϵ, Φ , such that there is an adjunction formula

$$K_X + B \sim_{\mathbb{R}} f^*(K_Z + B_Z + M_Z)$$

satisfying the following:

- $B_Z \in \Psi$, and
- $M_{Z'} = \sum_{i=1}^l r_i M_{i,Z'}$ on some high resolution $Z' \rightarrow Z$, where $r_i \in J$ and $M_{i,Z'}$ is Cartier nef for any i .

Therefore, we have

$$(Z, B_Z + M_Z) \in \mathcal{F}_{gklt}(d, J \cup \Psi, \leq v).$$

By Lemma 2.11, $(Z, B_Z + M_Z)$ is generalized δ -lc for some δ depending only on d, ϵ . Then Lemma 5.3 implies that

$$\text{Ivol}(K_X + B) = \text{vol}(K_Z + B_Z + M_Z)$$

is in a finite set depending only on d, δ, v, J, Ψ , hence depending only on d, ϵ, u, v, Φ . Without loss of generality, we can assume that $\text{Ivol}(K_X + B) = v_0$ for some fixed $v_0 \in \mathbb{R}^{>0}$. Therefore, $(X, B), A \in \mathcal{S}_{klt}(d, \Phi, \leq u, v_0)$, and the theorem follows from Theorem 1.5. \square

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